

DRINFELD ORBIFOLD ALGEBRAS FOR SYMMETRIC GROUPS

B. FOSTER-GREENWOOD AND C. KRILOFF

ABSTRACT. Drinfeld orbifold algebras are a type of deformation of skew group algebras generalizing graded Hecke algebras of interest in representation theory, algebraic combinatorics, and noncommutative geometry. In this article, we classify all Drinfeld orbifold algebras for symmetric groups acting by the natural permutation representation. This provides, for nonabelian groups, infinite families of examples of Drinfeld orbifold algebras that are not graded Hecke algebras. We include explicit descriptions of the maps recording commutator relations and show there is a one-parameter family of such maps supported only on the identity and a three-parameter family of maps supported only on 3-cycles and 5-cycles. Each commutator map must satisfy properties arising from a Poincaré-Birkhoff-Witt condition on the algebra, and our analysis of the properties illustrates reduction techniques using orbits of group element factorizations and intersections of fixed point spaces.

1. INTRODUCTION

Numerous algebras of intense recent study and interest arise as deformations of skew group algebras $S(V) \# G$, where G is a finite group acting linearly on a finite-dimensional vector space V and $S(V)$ is the symmetric algebra. A grading on the skew group algebra is determined by assigning degree one to vectors in V and degree zero to elements of the group algebra. Drinfeld graded Hecke algebras are constructed by identifying commutators of elements of V with carefully chosen elements of degree zero (i.e., from the group algebra) to yield a deformation of the skew group algebra. In [SW12a], Drinfeld orbifold algebras are similarly defined but additionally allow for degree-one terms in the commutator relations. The resulting algebras are also deformations of the skew group algebra.

Besides capturing a new realm of deformations of skew group algebras, Drinfeld orbifold algebras encompass many known algebras of interest in representation theory, noncommutative geometry, and mathematical physics. The term “Drinfeld orbifold algebras” alludes to the subject’s origins in [Dri86], where Drinfeld introduced a broad class of algebras to serve as noncommutative coordinate rings for singular orbifolds. When the group is a Coxeter group acting by its reflection representation, Drinfeld’s algebras are isomorphic (see [RS03]) to the graded Hecke algebras from [Lus88], which arise from a filtration of an affine Hecke algebra when the group is crystallographic (see [Lus89]). The representation theory of these algebras is useful in understanding representations and geometric structure of reductive p -adic groups.

2010 *Mathematics Subject Classification.* 16S80 (Primary) 16E40, 16S35, 20B30 (Secondary).

Key words and phrases. skew group algebra, deformations, Drinfeld orbifold algebra, Hochschild cohomology, Poincaré-Birkhoff-Witt conditions, symmetric group.

A recent focus on symplectic reflection algebras, which are Drinfeld Hecke algebras for symplectic reflection groups acting on a symplectic vector space, began with [EG02]. The importance of these algebras lies in the fact that the center of the skew group algebra is the ring of invariants, $\mathbb{C}[V]^G = \text{Spec}(V/G)$, and in the philosophy that the center of a deformation of the skew group algebra may then deform $\mathbb{C}[V]^G$ (see the surveys [Gor08, Bel16]). As a special case, rational Cherednik algebras arise by pairing a reflection representation with its dual and are related to integrable Calogero-Moser systems in physics and deep results in combinatorics (see for instance the surveys [Gor10, Eti14]).

Drinfeld orbifold algebras afford two advantageous views: as quotient algebras satisfying a Poincaré-Birkhoff-Witt (PBW) condition and as formal algebraic deformations of skew group algebras. While PBW conditions relate an algebra to homogeneous shadows of itself that have well-behaved bases, algebraic deformation theory (à la Gerstenhaber [GS88]) focuses on how the multiplicative structure varies with a deformation parameter and provides a framework of understanding via Hochschild cohomology. In particular, every formal deformation arises from a Hochschild 2-cocycle.

Fruitful techniques arise from a melding of the PBW perspective with the deformation theory perspective (see the survey [SW15]). Braverman and Gaitsgory [BG96] and also Polishchuk and Positelski [PP05] initiated the use of homological methods to study PBW conditions in the context of quadratic algebras of Koszul type. Etingof and Ginzburg applied some of these ideas in an expanded setting in their seminal paper on symplectic reflection algebras. The study of Drinfeld orbifold algebras also benefits from relating PBW conditions to formal deformations. Shepler and Witherspoon prove two characterizations of Drinfeld orbifold algebras: a concrete ring theoretic version [SW12a, Theorem 3.1] (proved using Composition-Diamond Lemmas and Groebner basis theory) and a cohomological version [SW12a, Theorem 7.2].

In the present case study, we classify Drinfeld orbifold algebras for symmetric groups acting by the natural permutation representation. In Section 4, we apply [SW12a, Theorem 7.2] and use Hochschild cohomology to find possible degree-one terms of the commutator relations for a Drinfeld orbifold algebra. In Section 7, we then work with [SW12a, Theorem 3.1] to determine compatible degree-zero terms (if they exist). Our main result, stated in Theorems 6.1 and 6.2, is an explicit description of the parameter maps that define Drinfeld orbifold algebras for symmetric groups.

Parameter maps of Drinfeld orbifold algebras record commutators of elements of the vector space V and can be categorized based on their support, i.e., which group elements appear in the image. Drinfeld orbifold algebra maps (see Definition 2.1) with their linear part supported only on the identity give rise to Lie orbifold algebras, as defined in [SW12a]. Lie orbifold algebras generalize universal enveloping algebras of Lie algebras, just as symplectic reflection algebras generalize Weyl algebras. We summarize our results classifying Lie and Drinfeld orbifold algebras.

Theorem. *For the symmetric group S_n ($n \geq 3$) acting on $V \cong \mathbb{C}^n$ by the natural permutation representation, there is a one-parameter family of Lie orbifold algebras.*

The remaining algebras have commutator relations supported only on 3-cycles and 5-cycles.

Theorem. *For the symmetric group S_n ($n \geq 4$) acting on $V \cong \mathbb{C}^n$ by the natural permutation representation, there is a three-parameter family of Drinfeld orbifold algebras supported on 3-cycles and 5-cycles. For $n = 3$, the family involves only two parameters.*

In Section 3, we present the algebras via generators and relations. The examples contribute to an expanding medley of “degree-one deformations”. For instance, Shakalli [Sha12] uses actions of Hopf algebras to construct examples of deformations of quantum skew group algebras involving degree-one terms in the commutator relations. Shepler and Witherspoon [SW12a] consider Drinfeld orbifold algebras for groups acting diagonally. The algebras we construct are among the first examples of Drinfeld orbifold (but not Hecke) algebras for nonabelian groups.

A fundamental problem in deformation theory is to determine which Hochschild 2-cocycles actually lift to deformations. The results of Section 6 provide a family of 2-cocycles that lift to define Drinfeld orbifold algebras for symmetric groups. However, we also show, in Proposition 6.3, that for symmetric groups, degree-one Hochschild 2-cocycles simultaneously supported on and off the identity do not lift to yield Drinfeld orbifold algebras (and in fact do not even define Poisson structures). This contrasts with the Drinfeld Hecke algebra case in which every polynomial degree-zero Hochschild 2-cocycle determines a deformation of the skew group algebra.

Reduction techniques in Section 5 and simplifications in Section 7 may prove helpful in predicting for which group actions and spaces candidate cocycles will lift to yield Drinfeld orbifold algebras. In particular, a variation of Lemma 5.4 may be effective for other groups with centralizers acting by monomial matrices, and the pattern to the values in Lemmas 7.1 and 7.6 might generalize to other group representations through analysis of intersections of fixed point spaces. As further exploration, one could consider Drinfeld orbifold algebras for symmetric groups in the twisted or quantum settings, as has been done for Drinfeld Hecke algebras for symmetric groups in [Wit07, Example 2.17] and [NW16, Theorem 6.9].

2. PRELIMINARIES

Throughout, we let G be a finite group acting linearly on a vector space $V \cong \mathbb{C}^n$. All tensors will be over \mathbb{C} .

Skew group algebras. Let G be a finite group that acts on a \mathbb{C} -algebra R by algebra automorphisms, and write ${}^g s$ for the result of acting by $g \in G$ on $s \in R$. The **skew group algebra** $R \# G$ is the semi-direct product algebra $R \rtimes \mathbb{C}G$ with underlying vector space $R \otimes \mathbb{C}G$ and multiplication of simple tensors defined by

$$(r \otimes g)(s \otimes h) = r({}^g s) \otimes gh$$

for all $r, s \in R$ and $g, h \in G$. The skew group algebra becomes a G -module by letting G act diagonally on $R \otimes \mathbb{C}G$, with conjugation on the group algebra factor:

$${}^g(s \otimes h) = ({}^g s) \otimes ({}^g h) = ({}^g s) \otimes ghg^{-1}.$$

In working with elements of skew group algebras, we commonly omit tensor symbols unless the tensor factors are lengthy expressions.

If G acts linearly on a vector space $V \cong \mathbb{C}^n$, then G also acts on the tensor algebra $T(V)$ and symmetric algebra $S(V)$ by algebra automorphisms. The skew group algebras $T(V) \# G$ and $S(V) \# G$ become graded algebras when elements of V are assigned degree one and elements of G are assigned degree zero.

Cochains. A k -**cochain** is a G -graded linear map $\alpha = \sum_{g \in G} \alpha_g g$ with components $\alpha_g : \bigwedge^k V \rightarrow S(V)$. (Details in Section 4 motivate the use of cohomological terminology.) If each α_g maps into V , then α is called a **linear cochain**, and if each α_g maps into \mathbb{C} , then α is called a **constant cochain**.

We regard a map α on $\bigwedge^k V$ as a multilinear alternating map on V^k and write $\alpha(v_1, \dots, v_k)$ in place of $\alpha(v_1 \wedge \dots \wedge v_k)$. Of course, if $\alpha(v_1, \dots, v_k) = 0$, then α is zero on any permutation of v_1, \dots, v_k . Also, if α is zero on all k -tuples of basis vectors, then α is zero on any k -tuple of vectors. We exploit these facts often in the computations in Section 7.

The **support of a cochain** α is the set of group elements for which the component α_g is not the zero map. The **kernel of a cochain** α is the set of vectors v_0 such that $\alpha(v_0, v_1, \dots, v_{k-1}) = 0$ for all $v_1, \dots, v_{k-1} \in V$.

The group G acts on the components of a cochain. Specifically, for a group element h and component α_g , the map ${}^h\alpha_g$ is defined by $({}^h\alpha_g)(v_1, \dots, v_k) = {}^h(\alpha_g({}^{h^{-1}}v_1, \dots, {}^{h^{-1}}v_k))$. In turn, the group acts on the space of cochains by letting ${}^h\alpha = \sum_{g \in G} {}^h\alpha_g \otimes hgh^{-1}$. Thus α is a **G -invariant cochain** if and only if ${}^h\alpha_g = \alpha_{hgh^{-1}}$ for all $g, h \in G$.

Drinfeld orbifold algebras. For a parameter map $\kappa = \kappa^L + \kappa^C$, where κ^L is a linear 2-cochain and κ^C is a constant 2-cochain, the quotient algebra

$$\mathcal{H}_\kappa = T(V) \# G / \langle vw - wv - \kappa^L(v, w) - \kappa^C(v, w) \mid v, w \in V \rangle$$

is called a **Drinfeld orbifold algebra** if the associated graded algebra $\text{gr } \mathcal{H}_\kappa$ is isomorphic to the skew group algebra $S(V) \# G$. The condition $\text{gr } \mathcal{H}_\kappa \cong S(V) \# G$ is called a **Poincaré-Birkhoff-Witt (PBW) condition**, in analogy with the PBW Theorem for universal enveloping algebras.

Further, if \mathcal{H}_κ is a Drinfeld orbifold algebra and t is a complex parameter, then

$$\mathcal{H}_{\kappa, t} := T(V) \# G[t] / \langle vw - wv - \kappa^L(v, w)t - \kappa^C(v, w)t^2 \mid v, w \in V \rangle$$

is called a **Drinfeld orbifold algebra over $\mathbb{C}[t]$** . In [SW12a, Theorem 2.1], Shepler and Witherspoon make an explicit connection between the PBW condition and deformations in the sense of Gerstenhaber [GS88] by showing how to interpret Drinfeld orbifold algebras over $\mathbb{C}[t]$ as formal deformations of the skew group algebra $S(V) \# G$. We summarize the broader context of formal deformations in Section 4.

Drinfeld orbifold algebra maps. Though the defining PBW condition for a Drinfeld orbifold algebra \mathcal{H}_κ involves an isomorphism of algebras, Shepler and Witherspoon proved an equivalent characterization [SW12a, Theorem 3.1] in terms of properties of the parameter map κ .

Definition 2.1. Let $\kappa = \kappa^L + \kappa^C$ where κ^L is a linear 2-cochain and κ^C is a constant 2-cochain, and let Alt_3 denote the alternating group on three elements. We say κ is a

Drinfeld orbifold algebra map if the following conditions are satisfied for all $g \in G$ and $v_1, v_2, v_3 \in V$:

$$(2.0) \quad \text{im } \kappa_g^L \subseteq V^g,$$

$$(2.1) \quad \text{the map } \kappa \text{ is } G\text{-invariant,}$$

$$(2.2) \quad \sum_{\sigma \in \text{Alt}_3} \kappa_g^L(v_{\sigma(2)}, v_{\sigma(3)})(^g v_{\sigma(1)} - v_{\sigma(1)}) = 0 \text{ in } S(V),$$

$$(2.3) \quad \sum_{\sigma \in \text{Alt}_3} \sum_{xy=g} \kappa_x^L(v_{\sigma(1)} + {}^y v_{\sigma(1)}, \kappa_y^L(v_{\sigma(2)}, v_{\sigma(3)})) = 2 \sum_{\sigma \in \text{Alt}_3} \kappa_g^C(v_{\sigma(2)}, v_{\sigma(3)})(^g v_{\sigma(1)} - v_{\sigma(1)}),$$

$$(2.4) \quad \sum_{\sigma \in \text{Alt}_3} \sum_{xy=g} \kappa_x^C(v_{\sigma(1)} + {}^y v_{\sigma(1)}, \kappa_y^L(v_{\sigma(2)}, v_{\sigma(3)})) = 0.$$

As a special case, if the linear component κ^L of a Drinfeld orbifold algebra map is supported only on the identity, then we also call κ a **Lie orbifold algebra map**.

Remark 2.2. If \mathcal{H}_κ is a Drinfeld orbifold algebra, then κ must satisfy conditions (2.1)-(2.4), but not necessarily the image constraint (2.0). However, [SW12a, Theorem 7.2 (ii)] guarantees there will exist a Drinfeld orbifold algebra $\mathcal{H}_{\tilde{\kappa}}$ such that $\mathcal{H}_{\tilde{\kappa}} \cong \mathcal{H}_\kappa$ as filtered algebras and $\tilde{\kappa}$ satisfies the image constraint $\text{im } \tilde{\kappa}_g^L \subseteq V^g$ for each g in G . Thus, in classifying Drinfeld orbifold algebras, it suffices to only consider Drinfeld orbifold algebra maps.

Theorem 2.3 ([SW12a, Theorem 3.1 and Theorem 7.2 (ii)]). *A quotient algebra \mathcal{H}_κ satisfies the PBW condition $\text{gr } \mathcal{H}_\kappa \cong S(V) \# G$ if and only if there exists a Drinfeld orbifold algebra map $\tilde{\kappa}$ such that $\mathcal{H}_\kappa \cong \mathcal{H}_{\tilde{\kappa}}$.*

The process of determining the set of all Drinfeld orbifold algebra maps consists of two phases. For reasons discussed in Section 4, we use language from cohomology and deformation theory to describe each phase. First, one finds all **pre-Drinfeld orbifold algebra maps**, i.e., all G -invariant linear 2-cochains κ^L satisfying Properties (2.0) and (2.2). A bijection between pre-Drinfeld orbifold algebra maps and a particular set of representatives of Hochschild cohomology classes facilitates this step (see Lemma 4.1). Second, we determine for which pre-Drinfeld orbifold algebra maps κ^L there exists a compatible G -invariant constant 2-cochain κ^C such that Properties (2.3) and (2.4) hold. We say κ^C **clears the first obstruction** if Property (2.3) holds and **clears the second obstruction** if Property (2.4) holds. If a G -invariant constant 2-cochain κ^C clears both obstructions, then we say κ^L **lifts** to the Drinfeld orbifold algebra map $\kappa = \kappa^L + \kappa^C$.

3. ORBIFOLD ALGEBRAS FOR SYMMETRIC GROUPS

Let e_1, \dots, e_n be the standard basis of $V \cong \mathbb{C}^n$. Let the symmetric group S_n act on V by its natural permutation representation, so ${}^\sigma e_i = e_{\sigma(i)}$ for σ in S_n . The main effort of this paper is in proving Theorems 6.1 and 6.2, which describe all Drinfeld orbifold algebra maps for S_n acting by the natural permutation representation. As corollaries of

the theorems in Section 6, we present here the resulting PBW deformations of the skew group algebra $S(V) \# S_n$ via generators and relations.

First, the one-dimensional space of Lie orbifold algebra maps classified in Theorem 6.1 yields a family of Lie orbifold algebras arising as deformations of $S(V) \# S_n$.

Theorem 3.1 (Lie Orbifold Algebras over $\mathbb{C}[t]$). *Let the symmetric group S_n ($n \geq 3$) act on $V \cong \mathbb{C}^n$ by its natural permutation representation. Then for $a \in \mathbb{C}$,*

$$\mathcal{H}_{\kappa,t} = T(V) \# S_n[t] / \langle e_i e_j - e_j e_i - a(e_i - e_j)t \mid 1 \leq i < j \leq n \rangle$$

is a Lie orbifold algebra over $\mathbb{C}[t]$. Further, the algebras $\mathcal{H}_{\kappa,1}$ are precisely the Drinfeld orbifold algebras such that κ^L is supported only on the identity.

Second, in Theorem 6.2, we determine (for S_n) all Drinfeld orbifold algebra maps such that the linear component κ^L is supported only off the identity. The relations in the consequent PBW deformations of $S(V) \# S_n$ involve sums of basis vectors over certain subsets of $[n] := \{1, \dots, n\}$. For $I \subseteq [n]$, let $e_I = \sum_{i \in I} e_i$, and let e_I^\perp denote the complementary vector $e_{[n]} - e_I$.

Theorem 3.2. *Let the symmetric group S_n ($n \geq 3$) act on $V \cong \mathbb{C}^n$ by its natural permutation representation. For $a, b, c \in \mathbb{C}$ and $1 \leq i < j \leq n$ let*

$$\kappa^L(e_i, e_j) = \sum_{k \neq i, j} (ae_{\{i, j, k\}} + be_{\{i, j, k\}}^\perp) \otimes ((ijk) - (kji)),$$

and let

$$\kappa^C(e_i, e_j) = c \sum_{k \neq i, j} ((ijk) - (kji)) + (a - b)^2 \left(\sum_{\substack{\sigma \text{ a 5-cycle} \\ \sigma^2(i)=j}} 2(\sigma - \sigma^{-1}) - \sum_{\substack{\sigma \text{ a 5-cycle} \\ \sigma(i)=j}} (\sigma - \sigma^{-1}) \right).$$

Then

$$\mathcal{H}_{\kappa,t} = T(V) \# S_n[t] / \langle e_i e_j - e_j e_i - \kappa^L(e_i, e_j)t - \kappa^C(e_i, e_j)t^2 \mid 1 \leq i < j \leq n \rangle$$

is a Drinfeld orbifold algebra over $\mathbb{C}[t]$. Further, the algebras $\mathcal{H}_{\kappa,1}$ are precisely the Drinfeld orbifold algebras such that $\text{im } \kappa_g^L \subseteq V^g$ for each $g \in S_n$ and κ^L is supported only off the identity.

We illustrate Theorem 3.2 for some small values of n . Note that the parameter b is irrelevant when $n = 3$, and the sums over 5-cycles are absent in the cases $n = 3$ and $n = 4$.

Example 3.3. For the symmetric group S_3 acting on $V \cong \mathbb{C}^3$ by the natural permutation representation, the Drinfeld orbifold algebras such that $\text{im } \kappa_g^L \subseteq V^g$ for each $g \in S_3$ and $\kappa_1^L = 0$ are the algebras of the form

$$\mathcal{H}_\kappa = T(V) \# S_3 / \langle e_i e_j - e_j e_i - \kappa(e_i, e_j) \mid 1 \leq i < j \leq 3 \rangle,$$

where for some $a, c \in \mathbb{C}$

$$\kappa(e_1, e_2) = \kappa(e_2, e_3) = \kappa(e_3, e_1) = (a(e_1 + e_2 + e_3) + c) \otimes ((123) - (321)).$$

This example coincides with [SW12a, Example 3.4] with a change of basis.

Example 3.4. For the symmetric group S_4 acting on $V \cong \mathbb{C}^4$ by the natural permutation representation, the Drinfeld orbifold algebras such that $\text{im } \kappa_g^L \subseteq V^g$ for each $g \in S_4$ and $\kappa_1^L = 0$ are the algebras of the form

$$\mathcal{H}_\kappa = T(V) \# S_4 / \langle e_i e_j - e_j e_i - \kappa(e_i, e_j) \mid 1 \leq i < j \leq 4 \rangle,$$

where

$$\begin{aligned} \kappa(e_1, e_2) = & (a(e_1 + e_2 + e_3) + b e_4 + c) \otimes ((123) - (321)) \\ & + (a(e_1 + e_2 + e_4) + b e_3 + c) \otimes ((124) - (421)), \end{aligned}$$

and $\kappa(e_{\sigma(1)}, e_{\sigma(2)}) = \sigma(\kappa(e_1, e_2))$ for σ in S_4 . (In acting by σ , recall that ${}^\sigma e_i = e_{\sigma(i)}$ and $\sigma\tau = \sigma\tau\sigma^{-1}$ for $\sigma, \tau \in S_n$.)

Example 3.5. For the symmetric group S_5 acting on $V \cong \mathbb{C}^5$ by the natural permutation representation, the Drinfeld orbifold algebras such that $\text{im } \kappa_g^L \subseteq V^g$ for each $g \in S_5$ and $\kappa_1^L = 0$ are the algebras of the form

$$\mathcal{H}_\kappa = T(V) \# S_5 / \langle e_i e_j - e_j e_i - \kappa(e_i, e_j) \mid 1 \leq i < j \leq 5 \rangle,$$

where

$$\begin{aligned} \kappa(e_1, e_2) = & (a(e_1 + e_2 + e_3) + b(e_4 + e_5) + c) \otimes ((123) - (321)) \\ & + (a(e_1 + e_2 + e_4) + b(e_5 + e_3) + c) \otimes ((124) - (421)) \\ & + (a(e_1 + e_2 + e_5) + b(e_3 + e_4) + c) \otimes ((125) - (521)) \\ & - (a - b)^2 \otimes ((12345) + (12543) + (12453) + (12354) + (12534) + (12435)) \\ & + (a - b)^2 \otimes ((21345) + (21543) + (21453) + (21354) + (21534) + (21435)) \\ & - 2(a - b)^2 \otimes ((23145) + (25143) + (24153) + (23154) + (25134) + (24135)) \\ & + 2(a - b)^2 \otimes ((13245) + (15243) + (14253) + (13254) + (15234) + (14235)), \end{aligned}$$

and $\kappa(e_{\sigma(1)}, e_{\sigma(2)}) = \sigma(\kappa(e_1, e_2))$ for σ in S_5 .

Remark 3.6. If we specialize to $t = 1$ and let $a = b = 0$ in Theorem 3.2, then the linear component κ^L is identically zero, thus recovering Drinfeld graded Hecke algebras for S_n .

4. DEFORMATION ALGEBRAS AND HOCHSCHILD COHOMOLOGY

Our goal in this section is to describe linear and constant 2-cochains κ that are G -invariant and satisfy the **mixed Jacobi identity**

$$[v_1, \kappa(v_2, v_3)] + [v_2, \kappa(v_3, v_1)] + [v_3, \kappa(v_1, v_2)] = 0 \quad \text{in } S(V) \# G.$$

When κ is expanded as $\sum_{g \in G} \kappa_g g$, it becomes clear that the mixed Jacobi identity for κ^L is equivalent to Property (2.2) of a Drinfeld orbifold algebra map, and the mixed Jacobi identity for κ^C is equivalent to Property (2.3) in the special case that the left side of (2.3) is zero. In light of the relation between Drinfeld orbifold algebras and formal deformations, Hochschild cohomology becomes a tool to facilitate finding Drinfeld orbifold algebra maps, as summarized in Lemma 4.1. We first review some background on

deformation theory and cohomology before turning to the specific case of the symmetric groups.

Deformations and Hochschild cohomology. Let A be an algebra over \mathbb{C} . For a complex parameter t , a **deformation over $\mathbb{C}[t]$** of A is the vector space $A[t]$ with an associative multiplication $*$, which is $\mathbb{C}[t]$ -bilinear and for a, b in A is recorded in the form

$$a * b = ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \cdots$$

for some maps $\mu_i : A \otimes A \rightarrow A$ with the sum finite for each pair a, b . Identifying coefficients on t^i in the expressions $a * (b * c)$ and $(a * b) * c$ yields a cohomological relation involving the maps μ_1, \dots, μ_i . For example, identifying coefficients of t shows that μ_1 is a **Hochschild 2-cocycle**, and identifying coefficients of t^2 shows the (Hochschild) coboundary of μ_2 must be half of the Gerstenhaber bracket of μ_1 with itself.

Generally, for an A -bimodule M the **Hochschild cohomology** of A with coefficients in M is $\mathrm{HH}^\bullet(A, M) := \mathrm{Ext}_{A \otimes A^{\mathrm{op}}}^\bullet(A, M)$, and if $M = A$, we simply write $\mathrm{HH}^\bullet(A)$. Hochschild cohomology may be computed using various resolutions, each with their own advantages. The maps μ_i defining the multiplication of a formal deformation algebra are most easily regarded as cochains on a bar resolution. However, when A is a skew group algebra, advantageous formulations of Hochschild cohomology arise from a Koszul resolution and frame cohomology in terms of invariant theory. Conversions between the bar complex and Koszul complex are key to the proof of [SW12a, Theorem 2.1] that shows how to interpret a Drinfeld orbifold algebra over $\mathbb{C}[t]$ as a formal deformation of a skew group algebra. The parameter map κ of a Drinfeld orbifold algebra $\mathcal{H}_{\kappa, t}$ over $\mathbb{C}[t]$ may be identified with a cochain on the Koszul complex, and the linear part κ^L relates to the first multiplication map μ_1 , while the constant part κ^C relates to the second multiplication map μ_2 (see [SW12a, Remark 2.5]).

Cohomological relations involving the maps μ_i have implications for the components of the parameter map κ . Indeed, the conditions on κ given in [SW12a, Theorem 3.1] have a parallel statement [SW12a, Theorem 7.2] in terms of cohomological spaces and operations. While Properties (2.1) and (2.2) of a Drinfeld orbifold algebra map are stated with minimal machinery, the cohomological interpretations aid in organizing computations and also reveal some hidden implications emphasized in Remark 4.2.

We now record descriptions of the cohomological spaces we use in our computations and refer the reader to [SW12a], for example, for more details on the bar and Koszul resolutions, chain maps, and isomorphisms that lead to these spaces. Let G be a finite group acting linearly on a vector space $V \cong \mathbb{C}^n$. Let H^\bullet be the G -graded vector space $H^\bullet = \bigoplus_{g \in G} H_g^\bullet$ with components

$$H_g^{p,d} = S^d(V^g) \otimes \bigwedge^{p - \mathrm{codim}(V^g)} (V^g)^* \otimes \bigwedge^{\mathrm{codim}(V^g)} ((V^g)^*)^\perp \otimes \mathbb{C}g,$$

where V^g is the fixed point space of g . Thus H^\bullet is tri-graded by cohomological degree p , homogeneous polynomial degree d , and group element g . For any set R carrying a G -action, we write R^G for the set of elements fixed by every g in G . With the group G acting diagonally on the tensor product (and with conjugation on the group algebra

factor), the Hochschild cohomology of $S(V)\#G$ can be computed using the series of isomorphisms

$$\mathrm{HH}^\bullet(S(V)\#G) \cong \mathrm{HH}^\bullet(S(V), S(V)\#G)^G \cong (H^\bullet)^G.$$

The first isomorphism follows from Štefan [Šte95] (for example), and the description of H^\bullet was first given independently by Farinati [Far05] and by Ginzburg-Kaledin [GK04].

Note that, together, the exterior factors of $H_g^{p,d}$ identify with a subspace of $\bigwedge^p V^*$, and then, since $S^d(V^g) \otimes \bigwedge^p V^* \otimes \mathbb{C}g \cong \mathrm{Hom}(\bigwedge^p V, S^d(V^g)g)$, the space H^\bullet may be identified with a subspace of the cochains introduced in Section 2. The next lemma records the relationship between Properties (2.1) and (2.2) of a Drinfeld orbifold algebra map and Hochschild cohomology. When $d = 1$, the lemma is a restatement of [SW12a, Theorem 7.2 (i) and (ii)]. When $d = 0$, the lemma is a restatement of [SW08, Corollary 8.17(ii)]. Despite its cohomological heritage, it is also possible to give a linear algebraic proof of Lemma 4.1 in the spirit of [RS03, Lemma 1.8].

Lemma 4.1. *For a 2-cochain $\kappa = \sum_{g \in G} \kappa_g g$ with $\mathrm{im} \kappa_g \subseteq S^d(V^g)$ for each $g \in G$, the following are equivalent:*

- (a) *The map κ is G -invariant and satisfies the mixed Jacobi identity, i.e., for all $v_1, v_2, v_3 \in V$*

$$[v_1, \kappa(v_2, v_3)] + [v_2, \kappa(v_3, v_1)] + [v_3, \kappa(v_1, v_2)] = 0 \quad \text{in } S(V)\#G,$$

where $[\cdot, \cdot]$ denotes the commutator in $S(V)\#G$.

- (b) *For all $g, h \in G$ and $v_1, v_2, v_3 \in V$:*

$$(i) \quad {}^h(\kappa_g(v_1, v_2)) = \kappa_{hgh^{-1}}({}^h v_1, {}^h v_2) \quad \text{and}$$

$$(ii) \quad \kappa_g(v_1, v_2)({}^g v_3 - v_3) + \kappa_g(v_2, v_3)({}^g v_1 - v_1) + \kappa_g(v_3, v_1)({}^g v_2 - v_2) = 0.$$

- (c) *The map κ is an element of*

$$(H^{2,d})^G = \left(\bigoplus_{g \in G} \left(S^d(V^g)g \otimes \bigwedge^{2-\mathrm{codim}(V^g)} (V^g)^* \otimes \bigwedge^{\mathrm{codim}(V^g)} ((V^g)^*)^\perp \right) \right)^G.$$

Remark 4.2. Part (c) of Lemma 4.1 illuminates some hidden implications of parts (a) and (b). For instance, κ can only be supported on elements g with $\mathrm{codim} V^g \in \{0, 2\}$, which is readily seen from part (c) by noting that negative exterior powers are zero and that an element g with codimension one acts nontrivially on $H_g^{2,d}$.

In practice, one may simplify the computation of $(H^\bullet)^G$ by computing centralizer invariants for a set of conjugacy class representatives and then expanding into G -invariants. Formally, $(H^\bullet)^G \cong \bigoplus_{g \in \mathcal{C}} (H_g^\bullet)^{Z(g)}$, where \mathcal{C} is a set of conjugacy class representatives, and $Z(g)$ is the centralizer of g . We review the explicit passage from a $Z(g)$ -invariant to a G -invariant, which will be especially relevant in translating the results of Lemma 4.5 into the maps in Definition 4.6. Recall that a cochain $\alpha = \sum_{g \in G} \alpha_g$ is G -invariant if and only if ${}^h \alpha_g = \alpha_{hgh^{-1}}$ for all $g, h \in G$. Thus, if α is a G -invariant cochain, then α_g is $Z(g)$ -invariant for each g , and α is determined by its components for a set of conjugacy class representatives. In particular, a centralizer invariant α_g extends uniquely to a G -invariant element, supported on the conjugacy class of g , via the map

$\alpha_g \mapsto \sum_{h \in [G/Z(g)]} {}^h \alpha_g$, where $[G/Z(g)]$ is a set of left coset representatives. Further, we have the following commutative diagram:

$$\begin{array}{ccc}
 \alpha_g \in (H_g^{2,d})^{Z(g)} & \longrightarrow & \alpha \in (H^{2,d})^G \\
 \downarrow & & \downarrow \\
 \kappa_g : \bigwedge^2 V \rightarrow S^d(V^g) & \longrightarrow & \kappa : \bigwedge^2 V \rightarrow \bigoplus_{h \in [G/Z(g)]} S^d(V^{hgh^{-1}})_{hgh^{-1}}.
 \end{array}$$

The vertical arrows are via the isomorphism $S^d(V^g)g \otimes \bigwedge^2 V^* \cong \text{Hom}(\bigwedge^2 V, S^d(V^g)g)$. The horizontal arrows are via the orbit-sum maps

$$\alpha_g \mapsto \sum_{h \in [G/Z(g)]} {}^h \alpha_g \quad \text{and} \quad \kappa_g \mapsto \sum_{h \in [G/Z(g)]} {}^h \kappa_g {}^h g.$$

Hochschild cohomology for symmetric groups. We now turn to the specific example of cohomology of skew group algebras of symmetric groups with the natural permutation representation. Much of the Hochschild cohomology of $S(V) \# S_n$ may be extracted as subcases of Hochschild cohomology for skew group algebras of complex reflection groups $G(r, p, n)$ found in [SW08]. However, we provide computations for $S_n \cong G(1, 1, n)$ here for purposes of self-containment and notational consistency.

For the remainder of the section, let e_1, \dots, e_n be the standard basis of $V \cong \mathbb{C}^n$, and let the symmetric group S_n act on V by its natural permutation representation. Thus for σ in S_n , we have $\sigma e_i = e_{\sigma(i)}$.

We first show that for the symmetric group acting by its natural permutation representation, elements of Hochschild 2-cohomology with polynomial degree zero or one can only be supported on the identity or 3-cycles.

Lemma 4.3. *Let S_n ($n \geq 3$) act on $V \cong \mathbb{C}^n$ by its natural permutation representation, and let $\alpha = \sum_{g \in S_n} \alpha_g$ be an element of $(H^{2,1} \oplus H^{2,0})^{S_n}$. If g is not the identity and not a 3-cycle, then $\alpha_g = 0$.*

Proof. Let $\alpha = \sum_{g \in S_n} \alpha_g$ be an element of the cohomology space $(H^{2,1} \oplus H^{2,0})^{S_n}$. If $\text{codim}(V^g) \notin \{0, 2\}$, then $\alpha_g = 0$ by Remark 4.2. Under the permutation representation, the only elements with $\text{codim } V^g \in \{0, 2\}$ are the identity, the 3-cycles, and the double-transpositions, so it remains to show that $\alpha_g = 0$ if g is a double-transposition. In fact, since α is determined by its components for a set of conjugacy class representatives, it suffices to show $(H_g^{2,1} \oplus H_g^{2,0})^{Z(g)} = 0$ for $g = (12)(34)$. The vectors

$$\begin{aligned}
 v_1 &= e_1 - e_2 - e_3 + e_4, \\
 v_2 &= e_1 - e_2 + e_3 - e_4, \\
 v_3 &= e_1 + e_2 - e_3 - e_4, \\
 v_4 &= e_1 + e_2 + e_3 + e_4, \\
 v_k &= e_k \quad \text{for } 5 \leq k \leq n
 \end{aligned}$$

form a g -eigenvector basis of V with $(V^g)^\perp = \text{Span}\{v_1, v_2\}$ and $V^g = \text{Span}\{v_3, \dots, v_n\}$. The wedge product $v_1^* \wedge v_2^*$ is a scalar multiple of the volume form

$$\text{vol}_g^\perp := e_1^* \wedge e_3^* + e_3^* \wedge e_2^* + e_2^* \wedge e_4^* + e_4^* \wedge e_1^*,$$

so vol_g^\perp is a basis for $\bigwedge^2((V^g)^\perp)^*$. The transposition (12) commutes with g but scales elements of $H_g^{2,1} \oplus H_g^{2,0}$ by negative one, so $(H_g^{2,1} \oplus H_g^{2,0})^{Z(g)} = 0$. \square

The cohomology elements in the next lemma give rise to the parameter maps of the Lie orbifold algebras exhibited in Theorem 3.1.

Lemma 4.4. *Let $G = S_n$ ($n \geq 3$) act on $V \cong \mathbb{C}^n$ by its natural permutation representation. The subspace of $(H^{2,1} \oplus H^{2,0})^{S_n}$ consisting of elements supported only on the identity is one-dimensional with basis $\sum_{1 \leq i < j \leq n} (e_i - e_j) \otimes e_i^* \wedge e_j^*$.*

Proof. Since the permutation representation of S_n is a self-dual reflection representation, generators of $(H_1^\bullet)^G \cong (S(V) \otimes \bigwedge V^*)^G \cong (S(V) \otimes \bigwedge V)^G$ are given by Solomon's Theorem from the invariant theory of reflection groups (see [Sol63], or the expository [Kan01, Chapter 22]). Specifically, the power sums $f_k = e_1^k + \dots + e_n^k$ with $1 \leq k \leq n$ form a set of algebraically independent invariant polynomials, and the differential forms

$$\alpha_k := \frac{1}{k+1} \sum_{i=1}^n \frac{\partial f_{k+1}}{\partial e_i} \otimes e_i^* = e_1^k \otimes e_1^* + \dots + e_n^k \otimes e_n^* \quad \text{for } 0 \leq k \leq n-1$$

generate $(S(V) \otimes \bigwedge V^*)^G$ as an exterior algebra over $S(V)^G \cong \mathbb{C}[f_1, \dots, f_n]$. Thus, $(S(V) \otimes \bigwedge^2 V^*)^G$ is freely generated as an $S(V)^G$ -module by $\{\alpha_k \alpha_l \mid 0 \leq k < l \leq n-1\}$. Since $\alpha_k \alpha_l$ has polynomial degree $k+l$, every element of $(H_1^{2,1} \oplus H_1^{2,0})^G$ is a scalar multiple of $\alpha_1 \alpha_0$. \square

The cohomology in the next lemma serves two purposes. The polynomial degree one elements give rise to pre-Drinfeld orbifold algebra maps supported only on 3-cycles, while the polynomial degree zero elements are needed in constructing multiple liftings of a pre-Drinfeld orbifold algebra map.

Lemma 4.5. *Let S_n ($n \geq 3$) act on $V \cong \mathbb{C}^n$ by its natural permutation representation. The subspace of $(H^{2,1} \oplus H^{2,0})^{S_n}$ consisting of elements supported only on 3-cycles is two-dimensional if $n = 3$ and three-dimensional if $n \geq 4$.*

Proof. Recall that a cohomology element is determined by its components for a set of conjugacy class representatives. Thus, if α is supported only on 3-cycles, it suffices to choose a representative, say $g = (123)$, and find a basis of

$$(H_g^{2,1} \oplus H_g^{2,0})^{Z(g)} \cong ((V^g \oplus \mathbb{C}) \otimes \bigwedge^2((V^g)^\perp)^* \otimes \mathbb{C}g)^{Z(g)}.$$

Let $\omega = e^{2\pi i/3}$. Then the vectors

$$\begin{aligned} v_1 &= e_1 + \omega^2 e_2 + \omega e_3, \\ v_2 &= e_1 + \omega e_2 + \omega^2 e_3, \\ v_3 &= e_1 + e_2 + e_3, \\ v_k &= e_k \quad \text{for } 4 \leq k \leq n \end{aligned}$$

form a g -eigenvector basis of V with $(V^g)^\perp = \text{Span}\{v_1, v_2\}$ and $V^g = \text{Span}\{v_3, \dots, v_n\}$. The wedge product $v_1^* \wedge v_2^*$ is a scalar multiple of the volume form

$$\text{vol}_g^\perp := e_1^* \wedge e_2^* + e_2^* \wedge e_3^* + e_3^* \wedge e_1^*,$$

so vol_g^\perp is a basis for $\bigwedge^2((V^g)^\perp)^*$. Each element of the centralizer

$$Z(123) = \langle (123) \rangle \times \text{Sym}_{\{4, \dots, n\}}$$

acts trivially on vol_g^\perp , so

$$(H_g^{2,1} \oplus H_g^{2,0})^{Z(g)} \cong ((V^g)^{Z(g)} \oplus \mathbb{C}) \otimes \bigwedge^2((V^g)^\perp)^* \otimes \mathbb{C}g.$$

The vectors $e_1 + e_2 + e_3$ and $e_4 + \dots + e_n$ form a basis of $(V^g)^{Z(g)}$, so the cohomology elements

$$\alpha_{(123)} = (e_1 + e_2 + e_3) \otimes \text{vol}_{(123)}^\perp \otimes (123) \quad \text{and} \quad \beta_{(123)} = (e_4 + \dots + e_n) \otimes \text{vol}_{(123)}^\perp \otimes (123)$$

span $(H_{(123)}^{2,1})^{Z(g)}$, while $\gamma_{(123)} = \text{vol}_g^\perp \otimes (123)$ spans $(H_{(123)}^{2,0})^{Z(g)}$. \square

The following definition arises from expanding the centralizer invariants determined in the proof of Lemma 4.5 into S_n -invariants and applying the isomorphism $S^d(V^g)g \otimes \bigwedge^2 V^* \cong \text{Hom}(\bigwedge^2 V, S^d(V^g)g)$. (See the discussion following Remark 4.2.)

Definition 4.6. For parameters $a, b \in \mathbb{C}$, let $\kappa_{3\text{-cyc}}^L = \sum_{(ijk) \in S_n} \kappa_{(ijk)}^L \otimes (ijk)$ be the linear 2-cochain with component maps $\kappa_{(ijk)}^L : \bigwedge^2 V \rightarrow V^{(ijk)}$ defined by

$$\kappa_{(ijk)}^L(e_i, e_j) = \kappa_{(ijk)}^L(e_j, e_k) = \kappa_{(ijk)}^L(e_k, e_i) = a(e_i + e_j + e_k) + b \sum_{l \neq i, j, k} e_l$$

and $\kappa_{3\text{-cyc}}^L(e_l, e_m) = 0$ if $\{e_l, e_m\} \cap V^{(ijk)} \neq \emptyset$.

For a parameter $c \in \mathbb{C}$, let $\kappa_{3\text{-cyc}}^C = \sum_{(ijk) \in S_n} \kappa_{(ijk)}^C \otimes (ijk)$ be the constant cochain with component maps $\kappa_{(ijk)}^C : \bigwedge^2 V \rightarrow \mathbb{C}$ defined by

$$\kappa_{(ijk)}^C(e_i, e_j) = \kappa_{(ijk)}^C(e_j, e_k) = \kappa_{(ijk)}^C(e_k, e_i) = c$$

and $\kappa_{3\text{-cyc}}^C(e_l, e_m) = 0$ if $\{e_l, e_m\} \cap V^{(ijk)} \neq \emptyset$.

Also let $\kappa_{3\text{-cyc}} = \kappa_{3\text{-cyc}}^L + \kappa_{3\text{-cyc}}^C$.

Notice that if a and b are not both zero, then $\ker \kappa_{(ijk)}^L = V^{(ijk)}$, and if $c \neq 0$, then $\ker \kappa_{(ijk)}^C = V^{(ijk)}$.

In view of the equivalences in Lemma 4.1, the polynomial degree one elements of Hochschild 2-cohomology computed in Lemmas 4.3, 4.4, and 4.5 yield a description of all pre-Drinfeld orbifold algebra maps.

Corollary 4.7. *The pre-Drinfeld orbifold algebra maps for S_n ($n \geq 3$) acting by its natural permutation representation are the linear 2-cochains $\kappa^L = \kappa_1^L + \kappa_{3\text{-cyc}}^L$, with $\kappa_{3\text{-cyc}}^L$ as in Definition 4.6 and $\kappa_1^L(e_i, e_j) = a_1(e_i - e_j)$ for some $a_1 \in \mathbb{C}$.*

In Theorems 6.1 and 6.2, we will show that the maps κ_1^L and $\kappa_{3\text{-cyc}}^L$ lift (separately, but not in combination) to Drinfeld orbifold algebra maps. Any two liftings of a particular pre-Drinfeld orbifold algebra map must differ by a constant 2-cochain that satisfies the mixed Jacobi identity. Recalling Lemma 4.1, the desired constant 2-cochains are revealed by the polynomial degree zero elements of Hochschild 2-cohomology computed in Lemmas 4.3, 4.4, and 4.5.

Corollary 4.8. *The S_n -invariant constant 2-cochains satisfying the mixed Jacobi identity are the maps $\kappa_{3\text{-cyc}}^C$ as in Definition 4.6.*

5. NOTATION AND REDUCTION TECHNIQUES

In this section, we gather notation and reduction techniques to facilitate the process of lifting pre-Drinfeld orbifold algebra maps κ^L to Drinfeld orbifold algebra maps $\kappa = \kappa^L + \kappa^C$. We first introduce operators on cochains to make it easier to refer to the properties of a Drinfeld orbifold algebra map (Definition 2.1) for a group G acting linearly on a vector space $V \cong \mathbb{C}^n$. Along the way, we indicate how our notation relates to the cohomological interpretation (see [SW12a, Theorem 7.2]) of each property. We then record symmetries that reduce the computations involved in clearing obstructions. We use these symmetries heavily in Section 7.

A variation on the coboundary. First, we define a map ψ to compactly describe the left-hand side of Property (2.2) and the right-hand side of Property (2.3) of a Drinfeld orbifold algebra map. For a linear or constant 2-cochain α , let $\psi(\alpha) = \sum_{g \in G} \psi_g g$ be the 3-cochain with components $\psi_g : \bigwedge^3 V \rightarrow S(V)$ given by

$$\psi_g(v_1, v_2, v_3) = \alpha_g(v_1, v_2)(^g v_3 - v_3) + \alpha_g(v_2, v_3)(^g v_1 - v_1) + \alpha_g(v_3, v_1)(^g v_2 - v_2).$$

The map ψ is the negation of the coboundary operator on cochains arising from the Koszul resolution. In particular, $\psi(\kappa^L) = -d_3^* \kappa^L$ and $\psi(\kappa^C) = -d_3^* \kappa^C$, where d_3^* is the coboundary operator that takes two-cochains to three-cochains (see the proof of [SW12a, Lemma 7.1]).

A variation on the cochain bracket. Next, we define a map ϕ to compactly describe the left-hand sides of Properties (2.3) and (2.4) of a Drinfeld orbifold algebra map. For α a linear or constant 2-cochain and β a linear 2-cochain, let $\phi(\alpha, \beta) = \sum_{g \in G} \phi_g g$ be the 3-cochain with components $\phi_g = \sum_{xy=g} \phi_{x,y}$, where $\phi_{x,y} : \bigwedge^3 V \rightarrow V \oplus \mathbb{C}$ is given by

$$\phi_{x,y}(v_1, v_2, v_3) = \alpha_x(v_1 + {}^y v_1, \beta_y(v_2, v_3)) + \alpha_x(v_2 + {}^y v_2, \beta_y(v_3, v_1)) + \alpha_x(v_3 + {}^y v_3, \beta_y(v_1, v_2)).$$

Thus $\phi(\alpha, \beta)$ is G -graded, with components ϕ_g , and also $(G \times G)$ -graded, with components $\phi_{x,y}$. The map $\phi(\alpha, \beta)$ is closely related to the cochain bracket $[\alpha, \beta]$ in [SW12a, Definition 5.6, Corollary 6.7]. Indeed, $\phi(\kappa^L, \kappa^L) = -\frac{1}{2}[\kappa^L, \kappa^L]$ and $\phi(\kappa^C, \kappa^L) = -[\kappa^C, \kappa^L]$, as explained in [SW12a, proof of Lemma 7.1].

To the extent possible, and especially in Section 7, we make remarks or calculations that apply to both $\phi(\kappa^L, \kappa^L)$ and $\phi(\kappa^C, \kappa^L)$, and in these instances, we write ϕ_g^* and $\phi_{x,y}^*$ for the corresponding components of $\phi(\kappa^*, \kappa^L)$ where $*$ denotes either L or C .

Drinfeld orbifold algebra maps (condensed definition). Equipped with the definitions of ψ and ϕ , the properties of a Drinfeld orbifold map $\kappa = \kappa^L + \kappa^C$ (Definition 2.1) may be expressed succinctly:

$$(2.0) \quad \text{im } \kappa_g^L \subseteq V^g \text{ for each } g \text{ in } G,$$

$$(2.1) \quad \text{the map } \kappa \text{ is } G\text{-invariant,}$$

$$(2.2) \quad \psi(\kappa^L) = 0,$$

$$(2.3) \quad \phi(\kappa^L, \kappa^L) = 2\psi(\kappa^C),$$

$$(2.4) \quad \phi(\kappa^C, \kappa^L) = 0.$$

Invariance relations. Recall that a cochain $\alpha = \sum_{g \in G} \alpha_g g$ with components $\alpha_g : \bigwedge^k V \rightarrow S(V)$ is G -invariant if and only if ${}^h \alpha_g = \alpha_{hgh^{-1}}$ for all $g, h \in G$. Equivalently,

$${}^h(\alpha_g(v_1, \dots, v_k)) = \alpha_{hgh^{-1}}({}^h v_1, \dots, {}^h v_k)$$

for all $g, h \in G$ and $v_1, \dots, v_k \in V$. Thus a G -invariant cochain is determined by its components for a set of conjugacy class representatives.

In the following lemma, one can let $\alpha = \kappa^L$ or $\alpha = \kappa^C$ and let $\beta = \kappa^L$ to see that if κ^L and κ^C are G -invariant, then $\phi(\kappa^*, \kappa^L)$ and $\psi(\kappa^*)$ are also G -invariant. This is helpful because, for instance, if $\phi_g = 2\psi_g$ for some $g \in G$, then acting by $h \in G$ on both sides shows $\phi_{hgh^{-1}} = 2\psi_{hgh^{-1}}$ also. Thus if $\phi_g = 2\psi_g$ for all g in a set of conjugacy class representatives, then $\phi(\kappa^L, \kappa^L) = 2\psi(\kappa^C)$. Similar reasoning applies to Properties (2.2) and (2.4) of a Drinfeld orbifold algebra map.

Lemma 5.1. *Let G be a finite group acting linearly on $V \cong \mathbb{C}^n$. If α and β are G -invariant 2-cochains with β linear and α linear or constant, then $\phi(\alpha, \beta)$ and $\psi(\alpha)$ are G -invariant. Specifically, at the component level, we have for all $g, h \in G$ and $v_1, v_2, v_3 \in V$*

$${}^h(\phi_{x,y}(v_1, v_2, v_3)) = \phi_{h x h^{-1}, h y h^{-1}}({}^h v_1, {}^h v_2, {}^h v_3),$$

$${}^h(\phi_g(v_1, v_2, v_3)) = \phi_{h g h^{-1}}({}^h v_1, {}^h v_2, {}^h v_3),$$

and

$${}^h(\psi_g(v_1, v_2, v_3)) = \psi_{h g h^{-1}}({}^h v_1, {}^h v_2, {}^h v_3).$$

Proof. To see that

$${}^h(\phi_{x,y}(v_1, v_2, v_3)) = \phi_{h x h^{-1}, h y h^{-1}}({}^h v_1, {}^h v_2, {}^h v_3)$$

for all $x, y, h \in G$ and $v_1, v_2, v_3 \in V$, note that, using invariance of α and β ,

$$\begin{aligned} {}^h(\alpha_x(v_i + {}^y v_i, \beta_y(v_j, v_k))) &= \alpha_{h x h^{-1}}({}^h v_i + {}^{h y} v_i, \beta_{h y h^{-1}}({}^h v_j, {}^h v_k)) \\ &= \alpha_{h x h^{-1}}({}^h v_i + {}^{h y h^{-1}}({}^h v_i), \beta_{h y h^{-1}}({}^h v_j, {}^h v_k)). \end{aligned}$$

Then also

$${}^h \phi_g = \sum_{xy=g} {}^h \phi_{x,y} = \sum_{xy=g} \phi_{h x h^{-1}, h y h^{-1}} = \phi_{h g h^{-1}},$$

where the last equality holds since the correspondence $(x, y) \leftrightarrow (h x h^{-1}, h y h^{-1})$ is a bijection between the set of factor pairs of g and the set of factor pairs of $h g h^{-1}$.

To see that ${}^h(\psi_g(v_1, v_2, v_3)) = \psi_{hgh^{-1}}({}^h v_1, {}^h v_2, {}^h v_3)$, again use that ${}^h(\alpha_g(v_i, v_j)) = \alpha_{hgh^{-1}}({}^h v_i, {}^h v_j)$ and ${}^{hg} v_k = {}^{hgh^{-1}}({}^h v_k)$. \square

Orbits of factorizations. The next observations involve the action of G on $G \times G$ by diagonal (componentwise) conjugation and provide a method for narrowing the number of terms and basis triples we must consider in evaluating $\phi(\kappa^*, \kappa^L)$ in Section 7.

If expressions $\phi_{x,y}(u, v, w)$ are organized in an array with rows indexed by factorizations xy of g and columns indexed by basis triples $\{u, v, w\}$, then $\phi_g(u, v, w)$ corresponds to a column sum. Our goal is to use invariance relations to show how to use column sums in a carefully chosen subarray to determine the column sums for the full array.

We first consider the effect of acting on a column sum for a subarray with rows indexed by factorizations in the same orbit under a subgroup.

Lemma 5.2. *Let G be a finite group acting linearly on $V \cong \mathbb{C}^n$, and let α and β be G -invariant 2-cochains with β linear and α linear or constant. Recall that $\phi(\alpha, \beta)$ has components $\phi_g = \sum_{xy=g} \phi_{x,y}$. Fix g in G , and let H be a subgroup of the centralizer $Z(g)$. If $g = xy$, then for all z in $Z(g)$ and u, v, w in V ,*

$$\begin{aligned} z \left(\sum_{(x', y') \in H(x, y)} \phi_{x', y'}(u, v, w) \right) &= \sum_{(x', y') \in H(x, y)} \phi_{z x', z y'}({}^z u, {}^z v, {}^z w) \\ &= \sum_{(x', y') \in {}^z H(x, y)} \phi_{x', y'}({}^z u, {}^z v, {}^z w). \end{aligned}$$

Proof. The first equality is an application of Lemma 5.1, and the second equality holds because the elements of ${}^H(x, y)$ and ${}^z H(x, y) = \{z(x', y') \mid (x', y') \in {}^H(x, y)\}$ are in bijection via the map $(x', y') \mapsto z(x', y')$. \square

We use the following characterization of when “coset orbits” of factorizations coincide to ensure there is no double-counting in the proof of Lemma 5.4.

Lemma 5.3. *Let G be a finite group, and let g be an element of G with factorization $g = xy$. Let $K = Z(x) \cap Z(y)$, the stabilizer of (x, y) under componentwise conjugation, and let H be a subgroup of $Z(g)$ normalized by K . Then the orbits ${}^{z_1 H}(x, y)$ and ${}^{z_2 H}(x, y)$ are disjoint or equal, with equality if and only if $z_1 H K = z_2 H K$.*

Proof. Let $z_1, z_2 \in Z(g)$ and suppose the orbits ${}^{z_1 H}(x, y)$ and ${}^{z_2 H}(x, y)$ intersect non-trivially so that ${}^{z_1 h_1}(x, y) = {}^{z_2 h_2}(x, y)$ for some h_1, h_2 in H . Then $h_1^{-1} z_1^{-1} z_2 h_2 \in K$, so $z_1^{-1} z_2 \in H K H = H K$ (since K normalizes H), and hence $z_1 H K = z_2 H K$. Then

$${}^{z_1 H}(x, y) = {}^{z_1 H K}(x, y) = {}^{z_2 H K}(x, y) = {}^{z_2 H}(x, y).$$

\square

The next lemma, stated in the specific case of the symmetric group, provides a method for using subgroups to reduce the number of expressions $\phi_{x,y}(e_i, e_j, e_k)$ that must be evaluated when verifying $\phi(\alpha, \beta) = 0$. Choosing the subgroup is a balancing act—using a small subgroup decreases the number of factorizations to consider but typically increases the number of basis triples, while using a large subgroup increases the number of factorizations to consider but decreases the number of basis triples.

Lemma 5.4. *Let S_n act on $V \cong \mathbb{C}^n$ by its natural permutation representation, and let α and β be S_n -invariant 2-cochains with β linear and α linear or constant. Recall that $\phi(\alpha, \beta)$ has components $\phi_g = \sum_{xy=g} \phi_{x,y}$. Suppose g is in S_n and has factorization $g = xy$. Let $K = Z(x) \cap Z(y)$, the stabilizer of (x, y) under componentwise conjugation, and let H be a subgroup of $Z(g)$ normalized by K . Let \mathcal{B} be the set of all three element subsets of $\{e_1, \dots, e_n\}$, and let \mathcal{B}_H be a set of H -orbit representatives of \mathcal{B} . If*

$$\sum_{(x', y') \in H(x, y)} \phi_{x', y'}(e_i, e_j, e_k) = 0 \quad \text{for all } \{e_i, e_j, e_k\} \in \mathcal{B}_H,$$

then

$$\sum_{(x', y') \in Z(g)(x, y)} \phi_{x', y'}(e_i, e_j, e_k) = 0 \quad \text{for all } \{e_i, e_j, e_k\} \in \mathcal{B}.$$

Proof. Use Lemma 5.2 with z ranging over the elements of H to show that

$$\sum_{(x', y') \in H(x, y)} \phi_{x', y'}(e_i, e_j, e_k) = 0 \quad \text{for all } \{e_i, e_j, e_k\} \in \mathcal{B}.$$

Then use Lemma 5.2 again to show for each z in $Z(g)$,

$$\sum_{(x', y') \in zH(x, y)} \phi_{x', y'}(e_i, e_j, e_k) = 0 \quad \text{for all } \{e_i, e_j, e_k\} \in \mathcal{B}.$$

Let $[Z(g)/HK]$ be a set of left coset representatives of HK , and use Lemma 5.3 to conclude,

$$\begin{aligned} & \sum_{(x', y') \in Z(g)(x, y)} \phi_{x', y'}(e_i, e_j, e_k) \\ &= \sum_{z \in [Z(g)/HK]} \left(\sum_{(x', y') \in zH(x, y)} \phi_{x', y'}(e_i, e_j, e_k) \right) = 0 \quad \text{for all } \{e_i, e_j, e_k\} \in \mathcal{B}. \end{aligned}$$

□

Remark 5.5. Though Lemma 5.4 is stated for the symmetric group, a similar idea might be useful in other groups where each centralizer acts by monomial matrices with respect to some basis.

6. LIFTING TO DEFORMATIONS

In Section 4 we determined all pre-Drinfeld orbifold algebra maps for the symmetric group acting by its natural permutation representation. When $n \geq 4$, the space of such maps is three-dimensional with one dimension of maps supported only on the identity and two dimensions of maps supported only on 3-cycles. We now determine for which candidate maps κ^L there exists a constant 2-cochain κ^C so that $\kappa = \kappa^L + \kappa^C$ also satisfies Properties (2.3) and (2.4) of a Drinfeld orbifold algebra map. We consider three cases, κ^L supported only on the identity (Theorem 6.1), supported only on 3-cycles (Theorem 6.2), and finally a combination supported on both the identity and 3-cycles (Proposition 6.3).

Lie orbifold algebra maps. The next theorem shows that for the symmetric group acting by its permutation representation, every pre-Drinfeld orbifold algebra map supported only on the identity lifts uniquely to a Lie orbifold algebra map. The corresponding Lie orbifold algebras are described in Theorem 3.1.

Theorem 6.1. *The Lie orbifold algebra maps for the symmetric group S_n ($n \geq 3$) acting on $V \cong \mathbb{C}^n$ by the natural permutation representation form a one-dimensional vector space generated by the map $\kappa : \bigwedge^2 V \rightarrow V \otimes \mathbb{C}S_n$ with $\kappa(e_i, e_j) = e_i - e_j$ for $1 \leq i < j \leq n$.*

Proof. Let κ^L be a pre-Drinfeld orbifold algebra map supported only on the identity. By Corollary 4.7, we have $\kappa^L(e_i, e_j) = a(e_i - e_j)$ for some $a \in \mathbb{C}$. It is straightforward to show that $\phi(\kappa^L, \kappa^L) = 0$, so now Property (2.3), $\phi(\kappa^L, \kappa^L) = 2\psi(\kappa^C)$, holds if and only if $\psi(\kappa^C) = 0$. By Corollary 4.8, the G -invariant constant 2-cochains such that $\psi(\kappa^C) = 0$ (i.e., satisfying the mixed Jacobi identity) are supported only on 3-cycles and have

$$\kappa_{(ijk)}^C(e_i, e_j) = \kappa_{(ijk)}^C(e_j, e_k) = \kappa_{(ijk)}^C(e_k, e_i) = c$$

for some scalar c .

Turning to Property (2.4), if $c = 0$, then $\kappa^C \equiv 0$, so $\phi(\kappa^C, \kappa^L) = 0$, and $\kappa = \kappa^L$ is a Lie orbifold algebra map. If $c \neq 0$, then $\kappa = \kappa^L + \kappa^C$ is not a Lie orbifold algebra map since $\phi(\kappa^C, \kappa^L) \neq 0$. In particular, the component $\phi_{(123)}$ of $\phi(\kappa^C, \kappa^L)$ is nonzero on the basis triple e_1, e_2, e_3 :

$$2[\kappa_{(123)}^C(e_1, e_2 - e_3) + \kappa_{(123)}^C(e_2, e_3 - e_1) + \kappa_{(123)}^C(e_3, e_1 - e_2)] = 12c \neq 0.$$

□

In general, a lifting need not be unique. See [SW12a, Example 4.3] for an example of a cyclic group having a Lie orbifold algebra map with κ^L and κ^C both nonzero.

Other Drinfeld orbifold algebra maps. Next we describe all possible Drinfeld orbifold algebra maps supported only off of the identity. We outline the proof here but relegate the details of clearing the obstructions to Section 7. The corresponding Drinfeld orbifold algebras are described in Theorem 3.2.

Theorem 6.2. *For S_n ($n \geq 3$) acting on $V \cong \mathbb{C}^n$ by its natural permutation representation, the Drinfeld orbifold algebra maps supported only off the identity are precisely the maps of the form $\kappa = \kappa_{3\text{-cyc}}^L + \kappa_{5\text{-cyc}}^C + \kappa_{3\text{-cyc}}^C$, with $\kappa_{3\text{-cyc}}$ as in Definition 4.6 and $\kappa_{5\text{-cyc}}^C$ as in Definition 7.4.*

Proof. Suppose κ^L is a pre-Drinfeld orbifold algebra map supported only off of the identity. By Corollary 4.7, we must have $\kappa^L = \kappa_{3\text{-cyc}}^L$ for some parameters $a, b \in \mathbb{C}$ as in Definition 4.6. Now, the goal is to find all G -invariant maps κ^C such that Properties (2.3) and (2.4) of a Drinfeld orbifold algebra map also hold.

First, we find a particular lifting.

- *First obstruction.* In Propositions 7.2 and 7.3, we evaluate $\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L)$. The results suggest Definition 7.4 of an S_n -invariant map $\kappa_{5\text{-cyc}}^C$ so that

$$\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L) = 2\psi(\kappa_{5\text{-cyc}}^C),$$

as verified in Proposition 7.5.

- *Second obstruction.* In Proposition 7.7, we show $\phi(\kappa_{5\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L) = 0$.

Thus $\kappa = \kappa_{3\text{-cyc}}^L + \kappa_{5\text{-cyc}}^C$ is a Drinfeld orbifold algebra map.

Next, we see how the particular lifting can be modified to produce all other possible liftings. Let κ^C be any G -invariant constant 2-cochain.

- *First obstruction.* Given that $\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L) = 2\psi(\kappa_{5\text{-cyc}}^C)$, we have that $\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L) = 2\psi(\kappa^C)$ if and only if $\psi(\kappa^C - \kappa_{5\text{-cyc}}^C) = 0$. By Corollary 4.8, $\psi(\kappa^C - \kappa_{5\text{-cyc}}^C) = 0$ if and only if $\kappa^C - \kappa_{5\text{-cyc}}^C = \kappa_{3\text{-cyc}}^C$, with $\kappa_{3\text{-cyc}}^C$ as in Definition 4.6 for some parameter $c \in \mathbb{C}$.
- *Second obstruction.* In Propositions 7.2 and 7.3, we show $\phi(\kappa_{3\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L) = 0$, and in Proposition 7.7, we show $\phi(\kappa_{5\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L) = 0$, so

$$\phi(\kappa_{3\text{-cyc}}^C + \kappa_{5\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L) = \phi(\kappa_{3\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L) + \phi(\kappa_{5\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L) = 0.$$

Thus the liftings of $\kappa_{3\text{-cyc}}^L$ to a Drinfeld orbifold algebra map are the maps of the form $\kappa = \kappa_{3\text{-cyc}}^L + \kappa_{3\text{-cyc}}^C + \kappa_{5\text{-cyc}}^C$. \square

Finally, we show for S_n that there are no Drinfeld orbifold algebra maps that are supported both on and off the identity.

Proposition 6.3. *Let κ^L be a pre-Drinfeld orbifold algebra map for the natural permutation representation of the symmetric group S_n ($n \geq 3$). If κ^L is supported both on the identity and off the identity, then κ^L does not lift to a Drinfeld orbifold algebra map.*

Proof. Let $\kappa^L \neq 0$ be a pre-Drinfeld orbifold algebra map. By Corollary 4.7, we have $\kappa^L = \kappa_1^L + \kappa_{3\text{-cyc}}^L$, where $\kappa_{3\text{-cyc}}^L$ is as in Definition 4.6 with parameters $a, b \in \mathbb{C}$, and $\kappa_1^L(e_i, e_j) = a_1(e_i - e_j)$ for some parameter $a_1 \in \mathbb{C}$. Suppose κ^C is an S_n -invariant constant 2-cochain. We show that if κ^C clears the first obstruction, i.e., $\phi(\kappa^L, \kappa^L) = 2\psi(\kappa^C)$, then $a = b = 0$ or $a_1 = 0$, so κ^L is supported only on the identity or only on the 3-cycles.

We let $g = (123)$ and compare the g -components of $2\psi(\kappa^C)$ and $\phi(\kappa^L, \kappa^L)$. First, note that since κ^C is S_n -invariant,

$$\kappa_{(123)}^C(e_1, e_2) = \kappa_{(123)}^C(e_2, e_3) = \kappa_{(123)}^C(e_3, e_1),$$

and so

$$\psi_{(123)}(e_1, e_2, e_3) = \kappa_{(123)}^C(e_1, e_2)(e_1 - e_3) + \kappa_{(123)}^C(e_2, e_3)(e_2 - e_1) + \kappa_{(123)}^C(e_3, e_1)(e_3 - e_2) = 0.$$

On the other hand,

$$\phi(\kappa^L, \kappa^L) = \phi(\kappa_1^L, \kappa_1^L) + \phi(\kappa_{3\text{-cyc}}^L, \kappa_1^L) + \phi(\kappa_1^L, \kappa_{3\text{-cyc}}^L) + \phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L).$$

The (123) -component of $\phi(\kappa_1^L, \kappa_1^L)$ is certainly zero, and by Proposition 7.2, the (123) -component of $\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L)$ is also zero. Only the cross terms remain, so

$$\phi_{(123)} = \phi_{(123),1} + \phi_{1,(123)}.$$

Recall

$$\phi_{x,y}(e_i, e_j, e_k) = \kappa_x^L[e_i + {}^y e_i, \kappa_y^L(e_j, e_k)] + \kappa_x^L[e_j + {}^y e_j, \kappa_y^L(e_k, e_i)] + \kappa_x^L[e_k + {}^y e_k, \kappa_y^L(e_i, e_j)].$$

The cross terms of $\phi_{(123)}$ evaluated on the basis triple e_1, e_2, e_3 are

$$\begin{aligned}\phi_{(123),1}(e_1, e_2, e_3) &= 2a_1(\kappa_{(123)}^L[e_1, e_2 - e_3] + \kappa_{(123)}^L[e_2, e_3 - e_1] + \kappa_{(123)}^L[e_3, e_1 - e_2]) \\ &= 4a_1(\kappa_{(123)}^L[e_1, e_2] + \kappa_{(123)}^L[e_2, e_3] + \kappa_{(123)}^L[e_3, e_1]) \\ &= 12a_1[a(e_1 + e_2 + e_3) + b(e_4 + \cdots + e_n)]\end{aligned}$$

and

$$\begin{aligned}\phi_{1,(123)}(e_1, e_2, e_3) &= 2\kappa_1^L[e_1 + e_2 + e_3, \kappa_{(123)}^L(e_1, e_2)] \\ &= 2a_1b[(n-3)(e_1 + e_2 + e_3) - 3(e_4 + \cdots + e_n)],\end{aligned}$$

which sum to

$$\phi_{(123)}(e_1, e_2, e_3) = 2a_1[(6a + (n-3)b)(e_1 + e_2 + e_3) + 3b(e_4 + \cdots + e_n)].$$

Thus, if $n \geq 4$, then $\phi_{(123)}(e_1, e_2, e_3) = 2\psi_{(123)}(e_1, e_2, e_3)$ if and only if $a_1 = 0$ or $b = 0 = a$. If $n = 3$, then $\kappa_{3\text{-cyc}}^L$ has only one parameter, a , and $\phi_{(123)}(e_1, e_2, e_3) = 12a_1a(e_1 + e_2 + e_3)$, which is zero if and only if $a_1 = 0$ or $a = 0$. \square

7. CLEARING THE OBSTRUCTIONS

In Section 4, we defined a pre-Drinfeld orbifold algebra map $\kappa_{3\text{-cyc}}^L$. This section is devoted to lifting $\kappa_{3\text{-cyc}}^L$ to a Drinfeld orbifold algebra map and provides the details of the proof of Theorem 6.2 outlined in Section 6. In view of Definition 2.1, we first evaluate $\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L)$ and “clear the first obstruction” by defining a G -invariant map $\kappa_{5\text{-cyc}}^C$ such that $\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L) = 2\psi(\kappa_{5\text{-cyc}}^C)$. Existence of such a map is predicted by [SW12b, Theorem 9.2]. We then “clear the second obstruction” by showing $\phi(\kappa_{5\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L) = 0$. These computations show that $\kappa = \kappa_{3\text{-cyc}}^L + \kappa_{5\text{-cyc}}^C$ is a Drinfeld orbifold algebra map.

Clearing the First Obstruction. We begin by recording simplifications of $\phi_{x,y}^*$, a summand of the component ϕ_g^* of $\phi(\kappa_{3\text{-cyc}}^*, \kappa_{3\text{-cyc}}^L)$, where $*$ stands for L or C . Simplification of $\phi_{x,y}^*(e_i, e_j, e_k)$ depends on the location of the basis vectors relative to the fixed spaces V^x and V^y , so we use the following indicator function. For $y \in S_n$ and $v \in V$, let

$$\delta_y(v) = \begin{cases} 1 & \text{if } v \in V^y \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7.1. *Let $\kappa_{3\text{-cyc}}^*$ with $*$ = L or $*$ = C be as in Definition 4.6 and let $\phi_{x,y}^*$ denote a term of the component ϕ_g^* of $\phi(\kappa_{3\text{-cyc}}^*, \kappa_{3\text{-cyc}}^L)$. Let x and y be 3-cycles such that $xy = g$, and let $1 \leq i, j, k \leq n$.*

- (1) *If $e_i, e_j \in V^y$, then $\phi_{x,y}^*(e_i, e_j, e_k) = 0$.*
- (2) *If $e_i \in V^y \cap V^x$, then $\phi_{x,y}^*(e_i, e_j, e_k) = 0$.*
- (3) *If $e_i \in V^y \setminus V^x$ and $e_j \notin V^y$, then*

$$\phi_{x,y}^*(e_i, e_j, {}^y e_j) = 2(b-a)[\delta_y({}^x e_i) - \delta_y({}^{x^{-1}} e_i)]\kappa_x^*[e_i, {}^x e_i].$$

(4) If $e_i \notin V^y$, then $\phi_{x,y}^*(e_i, {}^y e_i, {}^{y^2} e_i) = 0$.

Note that $\phi_{x,y}^*$ can be evaluated on any basis triple by using the alternating property along with these cases.

Proof. Consider

$$\phi_{x,y}^*(e_i, e_j, e_k) = \kappa_x^*[e_i + {}^y e_i, \kappa_y^L(e_j, e_k)] + \kappa_x^*[e_j + {}^y e_j, \kappa_y^L(e_k, e_i)] + \kappa_x^*[e_k + {}^y e_k, \kappa_y^L(e_i, e_j)].$$

Recall that if z is a 3-cycle then $V^z \subseteq \ker \kappa_z^*$.

- (1) If $e_i, e_j \in V^y$, then $\phi_{x,y}^*(e_i, e_j, e_k) = 0$ since $V^y \subseteq \ker \kappa_y^L$.
- (2) If $e_i \in V^y \cap V^x$, then the first term of $\phi_{x,y}^*$ vanishes because $e_i \in V^x \subseteq \ker \kappa_x^*$, and the second two terms of $\phi_{x,y}^*$ vanish because $e_i \in V^y \subseteq \ker \kappa_y^L$.
- (3) If $e_i \in V^y \setminus V^x$ and $e_j \notin V^y$, then $\phi_{x,y}^*(e_i, e_j, {}^y e_j) = 2\kappa_x^*[e_i, \kappa_y^L(e_j, {}^y e_j)]$. Using bilinearity and $V^x \subseteq \ker \kappa_x^*$, the right hand side is a linear combination of expressions $\kappa_x^*[e_i, {}^h e_i]$ for $h \in \langle x \rangle$. The appropriate coefficients, a or b , can be extracted using the fixed space indicator function. Also, $\sum_{h \in \langle x \rangle} {}^h e_i \in V^x \subseteq \ker \kappa_x^*$. Thus,

$$\begin{aligned} \phi_{x,y}^*(e_i, e_j, {}^y e_j) &= 2\kappa_x^*[e_i, \kappa_y^L(e_j, {}^y e_j)] \\ &= 2a \sum_{h \in \langle x \rangle} (1 - \delta_y({}^h e_i)) \kappa_x^*[e_i, {}^h e_i] + 2b \sum_{h \in \langle x \rangle} \delta_y({}^h e_i) \kappa_x^*[e_i, {}^h e_i] \\ &= 2(b - a) \sum_{h \in \langle x \rangle} \delta_y({}^h e_i) \kappa_x^*[e_i, {}^h e_i] \\ &= 2(b - a)[\delta_y({}^x e_i) - \delta_y({}^{x^{-1}} e_i)] \kappa_x^*[e_i, {}^x e_i]. \end{aligned}$$

- (4) Lastly, if $e_i \notin V^y$, note that $\phi_{x,y}^*(e_i, {}^y e_i, {}^{y^2} e_i) = 2\kappa_x^*[e_i + {}^y e_i + {}^{y^2} e_i, \kappa_y^L(e_i, {}^y e_i)]$ since $\kappa_y^L(e_i, {}^y e_i) = \kappa_y^L({}^y e_i, {}^{y^2} e_i) = \kappa_y^L({}^{y^2} e_i, e_i)$. Express $\kappa_y^L(e_i, {}^y e_i)$ as a linear combination of the vector $u = e_i + {}^y e_i + {}^{y^2} e_i$ and the vector $u_0 = e_1 + \dots + e_n$ (which is in the kernel of κ_x^*), to see that

$$\phi_{x,y}^*(e_i, {}^y e_i, {}^{y^2} e_i) = 2\kappa_x^*(u, (a - b)u + bu_0) = 0.$$

□

As mentioned in the outline of the proof of Theorem 6.2, the next two propositions are used to evaluate both $\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L)$ and $\phi(\kappa_{3\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L)$.

Proposition 7.2. *Let $\kappa_{3\text{-cyc}}^*$ with $*$ = L or $*$ = C be as in Definition 4.6. For $g \in S_n$, let ϕ_g^* be the g -component of $\phi(\kappa_{3\text{-cyc}}^*, \kappa_{3\text{-cyc}}^L)$. If g is not a 5-cycle then $\phi_g^* \equiv 0$.*

Proof. Since $\kappa_{3\text{-cyc}}^*$ is supported only on 3-cycles, we first determine the cycle types that arise as a product xy with x and y both 3-cycles. Since $\sigma^x \sigma^y = \sigma(xy)$, it suffices to examine representatives of orbits of factor pairs (x, y) under the action of S_n by diagonal conjugation. Orbit representatives and their products are

$$\begin{aligned} (123)(456), & \quad (123)(324) = (124), \\ (123)(345) = (12345), & \quad (123)(123) = (132), \\ (123)(234) = (12)(34), & \quad (123)(321) = 1. \end{aligned}$$

If the cycle type of g does not appear in this list, then certainly $\phi_g^* \equiv 0$. We leave the case $g = (12345)$ to Proposition 7.3 and show here that $\phi_g^* \equiv 0$ for $g = 1$, $g = (123)(456)$, $g = (12)(34)$, and $g = (123)$, and hence also for their conjugates.

Besides narrowing the set of representative elements g to consider, the list of orbit representatives reveals a way to organize the terms $\phi_{x,y}^*$ of ϕ_g^* . Specifically, if the cycle type of g occurs with multiplicity m in the list, then the factor pairs with product g are in m orbits under the diagonal conjugation action of $Z(g)$, and we can use a representative from each orbit to generate all the terms $\phi_{x,y}^*$ needed to evaluate ϕ_g^* .

Case 1 ($g = 1$). The identity component ϕ_1^* of $\phi(\kappa_{3\text{-cyc}}^*, \kappa_{3\text{-cyc}}^L)$ reduces to the sum of terms $\phi_{x,x^{-1}}^*$, where x ranges over the set of 3-cycles in S_n . For each 3-cycle x , we have $\text{im } \kappa_{x^{-1}}^L \subseteq V^{x^{-1}} = V^x \subseteq \ker \kappa_x^*$, so $\kappa_x^*[u, \kappa_{x^{-1}}^L(v, w)] = 0$ for all $u, v, w \in V$. It follows that $\phi_{x,x^{-1}}^* \equiv 0$ for each 3-cycle $x \in S_n$, and hence, $\phi_1^* \equiv 0$.

Case 2 ($g = (123)(456)$). If $g = (123)(456)$, then the component ϕ_g^* of $\phi(\kappa_{3\text{-cyc}}^*, \kappa_{3\text{-cyc}}^L)$ reduces to the sum of two terms, $\phi_{x,y}^* + \phi_{y,x}^*$, where $x = (123)$ and $y = (456)$. Note that $\text{im } \kappa_y^L \subseteq V^y \subseteq \ker \kappa_x^*$, so $\kappa_x^*[u, \kappa_y^L(v, w)] = 0$ for all $u, v, w \in V$, and the same holds if x and y are exchanged. It follows that both terms $\phi_{x,y}^*$ and $\phi_{y,x}^*$ are zero on any triple of vectors, and hence $\phi_g^* \equiv 0$.

Case 3 ($g = (12)(34)$). Note that $Z((12)(34)) = \langle (1324), (12) \rangle \times \text{Sym}_{\{5, \dots, n\}}$, and

$$\begin{aligned} \phi_g^* = \sum_{(x,y) \in Z(g)((123),(234))} \phi_{x,y}^* &= \phi_{(123),(234)}^* + \phi_{(342),(421)}^* + \phi_{(214),(143)}^* + \phi_{(431),(312)}^* \\ &\quad + \phi_{(213),(134)}^* + \phi_{(432),(321)}^* + \phi_{(124),(243)}^* + \phi_{(341),(412)}^*. \end{aligned}$$

Applying Lemma 5.4 with $H = \langle (1324) \rangle$, which is a normal subgroup of $Z(g)$, yields that to show $\phi_g^* \equiv 0$, it is sufficient to prove

$$\sum_{(x,y) \in H((123),(234))} \phi_{x,y}^*(e_i, e_j, e_k) = [\phi_{(123),(234)}^* + \phi_{(342),(421)}^* + \phi_{(214),(143)}^* + \phi_{(431),(312)}^*](e_i, e_j, e_k) = 0$$

for H -orbit representatives $\{e_i, e_j, e_k\}$ of the basis triples. Since $V^x \cap V^y \subseteq \ker \phi_{x,y}$ holds by Lemma 7.1 (2), we only consider $\{e_i, e_j, e_k\} \subseteq \{e_1, e_2, e_3, e_4\}$, and since the three element subsets of $\{e_1, e_2, e_3, e_4\}$ are in the same H -orbit, it suffices to show

$$[\phi_{(123),(234)}^* + \phi_{(342),(421)}^* + \phi_{(214),(143)}^* + \phi_{(431),(312)}^*](e_1, e_2, e_3) = 0.$$

The first three terms are zero by Lemma 7.1 (3) since $x e_i, x^{-1} e_i \notin V^y$ in all three cases and the fourth term is zero by Lemma 7.1 (4). Hence $\phi_{(12)(34)}^* \equiv 0$.

Case 4 ($g = (123)$). If $g = (123)$, then $Z((123)) = \langle (123) \rangle \times \text{Sym}_{\{4, \dots, n\}}$ and

$$\phi_g^* = \phi_{(132),(132)}^* + \sum_{r \in [n] \setminus [3]} \phi_{(12r),(r23)}^* + \phi_{(23r),(r31)}^* + \phi_{(31r),(r12)}^*.$$

We first consider the term $\phi_{(132),(132)}^*$ since the factorization $(132)(132)$ is in its own $Z(g)$ -orbit under diagonal conjugation. Note that $\text{im } \kappa_{(132)}^L \subseteq V^{(132)} \subseteq \ker \kappa_{(132)}^*$, so $\kappa_{(132)}^*(u, \kappa_{(132)}^L(v, w)) = 0$ for all $u, v, w \in V$, and thus, $\phi_{(132),(132)}^* \equiv 0$.

Applying Lemma 5.4 with $H = \langle (123) \rangle$, which is a normal subgroup of $Z(g)$, yields that in order to show $\sum_{r \in [n] \setminus [3]} \phi_{(12r), (r23)}^* + \phi_{(23r), (r31)}^* + \phi_{(31r), (r12)}^* = 0$, it is sufficient to prove

$$\sum_{(x,y) \in H((124), (423))} \phi_{x,y}^*(e_i, e_j, e_k) = [\phi_{(124), (423)}^* + \phi_{(234), (431)}^* + \phi_{(314), (412)}^*](e_i, e_j, e_k) = 0$$

for H -representatives $\{e_i, e_j, e_k\}$ of the basis triples. By Lemma 7.1 (2) $V^x \cap V^y \subseteq \ker \phi_{x,y}^*$, and hence we only consider $\{e_i, e_j, e_k\} \subseteq \{e_1, e_2, e_3, e_4\}$. The three element subsets of $\{e_1, e_2, e_3, e_4\}$ form two H -orbits, with representatives $\{e_1, e_2, e_k\}$ for $k = 3$ and $k = 4$. Note that

$$[\phi_{(124), (423)}^* + \phi_{(234), (431)}^* + \phi_{(314), (412)}^*](e_1, e_2, e_3) = 0$$

by Lemma 7.1 (3) applied to all three terms, noting that $x_{e_i}, x^{-1}_{e_i} \notin V^y$ in all cases. Also

$$[\phi_{(124), (423)}^* + \phi_{(234), (431)}^* + \phi_{(314), (412)}^*](e_1, e_2, e_4) = 0$$

by Lemma 7.1 (3) with $x_{e_i}, x^{-1}_{e_i} \notin V^y$ applied to the first two terms and Lemma 7.1 (4) applied to the third term. This verifies that $\phi_{(123)}^* \equiv 0$ and completes the proof. \square

Proposition 7.3. *Let $\kappa_{3\text{-cyc}} = \kappa_{3\text{-cyc}}^L + \kappa_{3\text{-cyc}}^C$ be as in Definition 4.6, with parameters $a, b, c \in \mathbb{C}$, and let ϕ_g^* denote the g -component of $\phi(\kappa_{3\text{-cyc}}^*, \kappa_{3\text{-cyc}}^L)$, where $*$ = L or $*$ = C and g is a 5-cycle. Then $\phi_g^* \equiv 0$. For ϕ_g^L , if $e_i \in V^g$, then $\phi_g^L(e_i, e_j, e_k) = 0$, and for $1 \leq i \leq n$,*

$$\begin{aligned} \phi_g^L(e_i, {}^g e_i, {}^{g^2} e_i) &= 2(a-b)^2(e_i + {}^g e_i - {}^{g^2} e_i - {}^{g^3} e_i) \text{ and} \\ \phi_g^L(e_i, {}^g e_i, {}^{g^3} e_i) &= 2(a-b)^2(-2e_i + 2{}^{g^2} e_i + {}^{g^3} e_i - {}^{g^4} e_i). \end{aligned}$$

Proof. It suffices to evaluate ϕ_g^* for the conjugacy class representative $g = (12345)$, since the results for any conjugate of g can be obtained by the orbit property described in Lemma 5.1. Note that $Z(g) = \langle (12345) \rangle \times \text{Sym}_{\{6, \dots, n\}}$, and as seen in the proof of Proposition 7.2, the factorizations of g as a product of 3-cycles are all in the same $Z(g)$ -orbit under diagonal conjugation, so

$$\phi_g^* = \phi_{(123), (345)}^* + \phi_{(234), (451)}^* + \phi_{(345), (512)}^* + \phi_{(451), (123)}^* + \phi_{(512), (234)}^*.$$

Note that for each pair of 3-cycles x and y with $xy = g = (12345)$ we have $V^g \subseteq V^x \cap V^y$, and $V^x \cap V^y \subseteq \ker \phi_{x,y}^*$ by Lemma 7.1 (2), so if any of the vectors in a basis triple lie in V^g then ϕ_g^* is zero on that triple. This leaves for further consideration only the cases where $\{e_i, e_j, e_k\} \subseteq \{e_1, e_2, e_3, e_4, e_5\}$.

First, consider $\phi_g^*(e_1, e_2, e_3)$. By Lemma 7.1 (1), the terms $\phi_{(123), (345)}^*(e_1, e_2, e_3)$ and $\phi_{(234), (451)}^*(e_1, e_2, e_3)$ are both zero. By Lemma 7.1 (4), the term $\phi_{(451), (123)}^*(e_1, e_2, e_3)$ is also zero. Applying Lemma 7.1 (3) to the remaining terms yields

$$\phi_g^*(e_1, e_2, e_3) = 2(a-b)(\kappa_{(512)}^*(e_1, e_2) - \kappa_{(345)}^*(e_3, e_4)).$$

Recall that $\kappa_{(ijk)}^C(e_i, e_j) = c$, $\kappa_{(ijk)}^L(e_i, e_j) = (a - b)(e_i + e_j + e_k) + b(e_1 + \dots + e_n)$, and that $e_I = \sum_{i \in I} e_i$ for $I \subseteq \{1, \dots, n\}$. Then

$$\begin{aligned} \phi_g^*(e_1, e_2, e_3) &= \begin{cases} 2(a - b)^2(e_{\{5,1,2\}} - e_{\{3,4,5\}}) & \text{if } * = L, \\ 0 & \text{if } * = C \end{cases} \\ &= \begin{cases} 2(a - b)^2(e_1 + e_2 - e_3 - e_4) & \text{if } * = L, \\ 0 & \text{if } * = C. \end{cases} \end{aligned}$$

Next, consider $\phi_g^*(e_1, e_2, e_4)$. By Lemma 7.1 (1), the term $\phi_{(123),(345)}^*(e_1, e_2, e_4)$ is zero. Using the alternating property to apply Lemma 7.1 (3) to the remaining terms yields

$$\begin{aligned} \phi_g^*(e_1, e_2, e_4) &= 2(a - b)(\kappa_{(234)}^*(e_2, e_3) + \kappa_{(345)}^*(e_4, e_5) - \kappa_{(451)}^*(e_4, e_5) - \kappa_{(512)}^*(e_1, e_2)) \\ &= \begin{cases} 2(a - b)^2(e_{\{2,3,4\}} + e_{\{3,4,5\}} - e_{\{4,5,1\}} - e_{\{5,1,2\}}) & \text{if } * = L, \\ 0 & \text{if } * = C \end{cases} \\ &= \begin{cases} 2(a - b)^2(-2e_1 + 2e_3 + e_4 - e_5) & \text{if } * = L, \\ 0 & \text{if } * = C. \end{cases} \end{aligned}$$

Finally, note that for any $e_i \notin V^g$, the values of $\phi_g^*(e_i, {}^g e_i, {}^{g^2} e_i)$ and $\phi_g^*(e_i, {}^g e_i, {}^{g^3} e_i)$ are obtained from the cases $\phi_g^*(e_1, e_2, e_3)$ and $\phi_g^*(e_1, e_2, e_4)$ by acting by an appropriate power of g and using the orbit property in Lemma 5.1. \square

The next definition of a map $\kappa_{5\text{-cyc}}^C$ supported only on 5-cycles is motivated by the requirement $\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L) = 2\psi(\kappa^C)$. When G is S_3 or S_4 there are no 5-cycles and $\kappa_{5\text{-cyc}}^C$ is the zero map.

Definition 7.4. For parameters $a, b \in \mathbb{C}$, define an S_n -invariant map $\kappa_{5\text{-cyc}}^C = \sum_{g \in S_n} \kappa_g^C g$ with component maps $\kappa_g^C : \bigwedge^2 V \rightarrow \mathbb{C}$. If g is not a 5-cycle, let $\kappa_g^C \equiv 0$. If g is a 5-cycle, define κ_g^C by the skew-symmetric matrix

$$[\kappa_g^C] = (a - b)^2([g] - [g]^T - 2[g^2] + 2[g^2]^T),$$

where $[g]$ denotes the matrix of g with respect to the basis e_1, \dots, e_n , and the (i, j) -entry of $[\kappa_g^C]$ records $\kappa_g^C(e_i, e_j)$.

In practice we use the consequences that $V^g \subseteq \ker \kappa_g^C$, and if $e_i \notin V^g$, then

$$\kappa_g^C(e_i, {}^g e_i) = -(a - b)^2 \quad \text{and} \quad \kappa_g^C(e_i, {}^{g^2} e_i) = 2(a - b)^2.$$

Also, $\kappa_{5\text{-cyc}}^C$ is G -invariant, i.e., $\kappa_{hgh^{-1}}^C({}^h e_i, {}^h e_j) = \kappa_g^C(e_i, e_j)$ for all $h, g \in S_n$ and all $1 \leq i, j \leq n$.

Proposition 7.5. Let $\kappa_{3\text{-cyc}}^L$ and $\kappa_{5\text{-cyc}}^C$ be as in Definitions 4.6 and 7.4, with common parameters $a, b \in \mathbb{C}$. Then $\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L) = 2\psi(\kappa_{5\text{-cyc}}^C)$.

Proof. We compare the component ϕ_g of $\phi(\kappa_{3\text{-cyc}}^L, \kappa_{3\text{-cyc}}^L)$ with the component $2\psi_g$ of $2\psi(\kappa_{5\text{-cyc}}^C)$. If g is not a 5-cycle, then $\phi_g \equiv 0$ by Proposition 7.2; and $\kappa_{5\text{-cyc}}^C$ is not supported on g , so $2\psi_g \equiv 0$ also. If g is a 5-cycle, then note that the results of Proposition 7.3 can be written in the form

$$\begin{aligned}\phi_g(e_i, {}^g e_i, {}^{g^2} e_i) &= -2(a-b)^2({}^g e_i - e_i) + 2({}^{g^2} e_i - {}^g e_i) + ({}^{g^3} e_i - {}^{g^2} e_i), \\ \phi_g(e_i, {}^g e_i, {}^{g^3} e_i) &= 2(a-b)^2(2({}^g e_i - e_i) + 2({}^{g^2} e_i - {}^g e_i) - ({}^{g^4} e_i - {}^{g^3} e_i)),\end{aligned}$$

while

$$\begin{aligned}\psi_g(e_i, {}^g e_i, {}^{g^2} e_i) &= \kappa_g^C({}^g e_i, {}^{g^2} e_i)({}^g e_i - e_i) + \kappa_g^C({}^{g^2} e_i, e_i)({}^{g^2} e_i - {}^g e_i) + \kappa_g^C(e_i, {}^g e_i)({}^{g^3} e_i - {}^{g^2} e_i), \\ \psi_g(e_i, {}^g e_i, {}^{g^3} e_i) &= \kappa_g^C({}^g e_i, {}^{g^3} e_i)({}^g e_i - e_i) + \kappa_g^C({}^{g^3} e_i, e_i)({}^{g^2} e_i - {}^g e_i) + \kappa_g^C(e_i, {}^g e_i)({}^{g^4} e_i - {}^{g^3} e_i).\end{aligned}$$

Finally, if $e_i \in V^g$, then $\phi_g(e_i, e_j, e_k) = 0$ and

$$\psi_g(e_i, e_j, e_k) = \kappa_g^C(e_j, e_k)({}^g e_i - e_i) + \kappa_g^C(e_k, e_i)({}^g e_j - e_j) + \kappa_g^C(e_i, e_j)({}^g e_k - e_k) = 0,$$

where the first term vanishes because ${}^g e_i - e_i = 0$ and the second two terms vanish because $V^g \subseteq \ker \kappa_g^C$. Since ψ is alternating, we see that $\psi_g(e_i, e_j, e_k) = 0$ whenever $\{e_i, e_j, e_k\} \cap V^g \neq \emptyset$. \square

Clearing the Second Obstruction. The final step in showing $\kappa = \kappa_{3\text{-cyc}}^L + \kappa_{5\text{-cyc}}^C$ is a Drinfeld orbifold algebra map is to verify $\phi(\kappa_{5\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L) = 0$.

We begin with a lemma that describes simplifications of the summands $\phi_{x,y}$ of the components ϕ_g of $\phi(\kappa_{5\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L)$. As in the analogous Lemma 7.1, simplification of $\phi_{x,y}(e_i, e_j, e_k)$ depends on where the vectors in the basis triple lie relative to the fixed spaces V^x and V^y . Recall that for $\sigma \in S_n$ and $v \in V$, $\delta_\sigma(v) = 1$ if $v \in V^\sigma$ and $\delta_\sigma(v) = 0$ otherwise.

Lemma 7.6. *Let $\kappa_{5\text{-cyc}}^C$ and $\kappa_{3\text{-cyc}}^L$ be as in Definitions 7.4 and 4.6, with common parameters $a, b \in \mathbb{C}$. Let $\phi_{x,y}$ denote a summand of the component ϕ_g of $\phi(\kappa_{5\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L)$. Let x be a 5-cycle and y be a 3-cycle. Let e_i, e_j, e_k be basis vectors.*

- (1) *If $e_i, e_j \in V^y$, then $\phi_{x,y}(e_i, e_j, e_k) = 0$.*
- (2) *If $e_i \in V^y \cap V^x$, then $\phi_{x,y}(e_i, e_j, e_k) = 0$.*
- (3) *If $e_i \in V^y \setminus V^x$ and $e_j \notin V^y$, then*

$$\phi_{x,y}(e_i, e_j, {}^y e_j) = 2(a-b)^3 \left[\delta_y({}^x e_i) - 2\delta_y({}^{x^2} e_i) + 2\delta_y({}^{x^{-2}} e_i) - \delta_y({}^{x^{-1}} e_i) \right].$$

- (4) *If $e_i \notin V^y$, then $\phi_{x,y}(e_i, {}^y e_i, {}^{y^2} e_i) = 0$.*

Note that $\phi_{x,y}$ can be evaluated on any basis triple by using the alternating property along with these cases.

Proof. The proofs of parts (1), (2), and (4) are the same as in the proof of Lemma 7.1 since $V^y \subseteq \ker \kappa_y^L$, and $V^x \subseteq \ker \kappa_x^C$ is also true for $\kappa_{5\text{-cyc}}^C$. The proof of part (3) is the

same up until the last step, so if $e_i \in V^y \setminus V^x$ and $e_j \notin V^y$, then

$$\begin{aligned} \phi_{x,y}(e_i, e_j, {}^y e_j) &= 2(b-a) \sum_{h \in \langle x \rangle} \delta_y({}^h e_i) \kappa_x^C[e_i, {}^h e_i] \\ &= 2(a-b)^3 \left[\delta_y({}^x e_i) - 2\delta_y({}^{x^2} e_i) + 2\delta_y({}^{x^{-2}} e_i) - \delta_y({}^{x^{-1}} e_i) \right]. \end{aligned}$$

□

The proof of the next proposition uses these simplifications to verify that indeed $\phi(\kappa_{5\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L) = 0$, as mentioned in the outline of the proof of Theorem 3.2. This clears the second obstruction.

Proposition 7.7. *Let $\kappa_{5\text{-cyc}}^C$ and $\kappa_{3\text{-cyc}}^L$ be as in Definitions 7.4 and 4.6, with common parameters $a, b \in \mathbb{C}$. For every $g \in S_n$, the component ϕ_g of $\phi(\kappa_{5\text{-cyc}}^C, \kappa_{3\text{-cyc}}^L)$ is identically zero.*

Proof. Since $\kappa_{5\text{-cyc}}^C$ is supported only on 5-cycles and $\kappa_{3\text{-cyc}}^L$ is supported only on 3-cycles, we first determine the cycle types that arise as a product xy with x a 5-cycle and y a 3-cycle. Since ${}^\sigma x {}^\sigma y = {}^\sigma(xy)$, it suffices to examine representatives of orbits of factor pairs (x, y) under the action of S_n by diagonal conjugation. Orbit representatives and their products are

$$\begin{aligned} (12345)(678), & & (12345)(567) &= (1234567), \\ (12345)(456) &= (1234)(56), & (12345)(546) &= (12346), \\ (12345)(356) &= (123)(456), & (12345)(536) &= (1236)(45), \\ (12345)(345) &= (12354), & (12345)(543) &= (123), \\ (12345)(245) &= (12534), & (12345)(542) &= (12)(34). \end{aligned}$$

If the cycle type of g does not appear in this list, then certainly $\phi_g \equiv 0$. We show further that $\phi_g \equiv 0$ for $g = (12345)(678)$, $g = (1234567)$, $g = (1234)(56)$, $g = (123)(456)$, $g = (12345)$, $g = (12)(34)$, and $g = (123)$, and hence also for their conjugates.

Besides narrowing the set of representative elements g to consider, the list of orbit representatives reveals a way to organize the terms $\phi_{x,y}$ of ϕ_g . Specifically, if the cycle type of g occurs with multiplicity m in the list, then the factor pairs with product g are in m orbits under the diagonal conjugation action of $Z(g)$, and we can use a representative from each orbit to generate all the terms $\phi_{x,y}$ needed to evaluate ϕ_g .

Case 1 ($g = (12345)(678)$). Note that g has a unique factorization as a product of a 5-cycle and a 3-cycle. To show $\phi_g = \phi_{(12345)(678)} \equiv 0$, we apply Lemma 5.4 with $H = Z(g) = \langle (12345), (678) \rangle \times \text{Sym}_{\{9, \dots, n\}}$. In particular, it suffices to evaluate ϕ_g on $Z(g)$ -orbit representatives of the basis triples. Since $V^x \cap V^y \subseteq \ker \phi_{x,y}$, we only consider $\{e_i, e_j, e_k\} \subseteq \{e_l \mid 1 \leq l \leq 8\}$. If $\{e_i, e_j, e_k\}$ does not contain at least two elements from $\{e_6, e_7, e_8\}$, then $\phi_{x,y}(e_i, e_j, e_k) = 0$ by Lemma 7.6 (1). The remaining triples partition into $Z(g)$ -orbits

$$Z(g)\{e_1, e_6, e_7\} \quad \text{and} \quad Z(g)\{e_6, e_7, e_8\}.$$

Note that $\phi_{x,y}(e_1, e_6, e_7) = 0$ by Lemma 7.6 (3) and $\phi_{x,y}(e_6, e_7, e_8) = 0$ by Lemma 7.6 (4).

Case 2 ($g = (1234567)$). Note that $Z(g) = \langle (1234567) \rangle \times \text{Sym}_{\{8, \dots, n\}}$. The factorizations of g as a product of a 5-cycle and a 3-cycle are in a single $Z(g)$ -orbit. To show $\phi_g \equiv 0$, it suffices to verify

$$\sum_{(x,y) \in Z(g)((12345), (567))} \phi_{x,y}(e_i, e_j, e_k) = 0$$

for $Z(g)$ -orbit representatives $\{e_i, e_j, e_k\}$ of the basis triples. Since $V^x \cap V^y \subseteq \ker \phi_{x,y}$, we only consider $\{e_i, e_j, e_k\} \subseteq \{e_l \mid 1 \leq l \leq 7\}$. The remaining triples partition into $Z(g)$ -orbits

$$Z(g)\{e_1, e_2, e_3\}, \quad Z(g)\{e_1, e_2, e_4\}, \quad Z(g)\{e_1, e_2, e_5\}, \quad Z(g)\{e_1, e_2, e_6\}, \quad \text{and} \quad Z(g)\{e_1, e_3, e_5\}.$$

(The sum of orbit sizes is $7 + 7 + 7 + 7 + 7 = \binom{7}{3}$.) We simplify each remaining term $\phi_{x,y}(e_i, e_j, e_k)$ of the sum and record the results in a table with rows indexed by the elements of $Z(g)((12345), (567))$. Notice $\phi_{(45671), (123)}(e_1, e_2, e_3) = 0$ by Lemma 7.6 (4), all other zero entries result from Lemma 7.6 (1), and all remaining entries are found using Lemma 7.6 (3). Each column sum is zero.

x, y	$\phi_{x,y}(e_1, e_2, e_3)$	$\phi_{x,y}(e_1, e_2, e_4)$	$\phi_{x,y}(e_1, e_2, e_5)$	$\phi_{x,y}(e_1, e_2, e_6)$	$\phi_{x,y}(e_1, e_3, e_5)$
(12345), (567)	0	0	0	0	0
(23456), (671)	0	0	0	$-2(a-b)^3$	0
(34567), (712)	$2(a-b)^3$	$-4(a-b)^3$	$4(a-b)^3$	$-2(a-b)^3$	0
(45671), (123)	0	$2(a-b)^3$	$-4(a-b)^3$	$4(a-b)^3$	$4(a-b)^3$
(56712), (234)	$-2(a-b)^3$	$2(a-b)^3$	0	0	0
(67123), (345)	0	0	0	0	$-4(a-b)^3$
(71234), (456)	0	0	0	0	0

Case 3 ($g = (1234)(56)$). Note that $Z(g) = \langle (1234), (56) \rangle \times \text{Sym}_{\{7, \dots, n\}}$. The factorizations of g as a product of a 5-cycle and a 3-cycle are in two $Z(g)$ -orbits

$$Z(g)((12345), (564)) \quad \text{and} \quad Z(g)((12356), (634)).$$

To show $\phi_g \equiv 0$, it suffices to verify

$$\sum_{(x,y) \in Z(g)((12345), (564))} \phi_{x,y}(e_i, e_j, e_k) + \sum_{(x,y) \in Z(g)((12356), (634))} \phi_{x,y}(e_i, e_j, e_k) = 0$$

for $Z(g)$ -orbit representatives $\{e_i, e_j, e_k\}$ of the basis triples. Since $V^x \cap V^y \subseteq \ker \phi_{x,y}$, we only consider $\{e_i, e_j, e_k\} \subseteq \{e_l \mid 1 \leq l \leq 6\}$. The remaining triples partition into $Z(g)$ -orbits

$$Z(g)\{e_1, e_2, e_3\}, \quad Z(g)\{e_1, e_2, e_5\}, \quad Z(g)\{e_1, e_3, e_5\}, \quad \text{and} \quad Z(g)\{e_1, e_5, e_6\}.$$

(The sum of orbit sizes is $4 + 8 + 4 + 4 = \binom{6}{3}$.) We simplify each term $\phi_{x,y}(e_i, e_j, e_k)$ of the sum and record the results in a table with rows indexed by the elements of $Z(g)((12345), (564)) \cup Z(g)((12356), (634))$. Each zero entry is color-coded and tagged by

the applicable part of Lemma 7.6, all nonzero entries are found using Lemma 7.6 (3), and each column sum is zero.

x, y	$\phi_{x,y}(e_1, e_2, e_3)$	$\phi_{x,y}(e_1, e_2, e_5)$	$\phi_{x,y}(e_1, e_3, e_5)$	$\phi_{x,y}(e_1, e_5, e_6)$
(12345), (456)	0	0	0	$-2(a-b)^3$
(23415), (156)	0	$2(a-b)^3$	0	0
(34125), (256)	0	$2(a-b)^3$	0	$2(a-b)^3$
(41235), (356)	0	0	0	0
(12346), (465)	0	0	0	$2(a-b)^3$
(23416), (165)	0	$-2(a-b)^3$	0	0
(34126), (265)	0	$-2(a-b)^3$	0	$-2(a-b)^3$
(41236), (365)	0	0	0	0
(12356), (346)	0	0	0	0
(23456), (416)	0	0	0	0
(34156), (126)	$6(a-b)^3$	0	0	0
(41256), (236)	$-6(a-b)^3$	0	0	0
(12365), (345)	0	0	$-6(a-b)^3$	0
(23465), (415)	0	$-6(a-b)^3$	$6(a-b)^3$	0
(34165), (125)	$6(a-b)^3$	0	$6(a-b)^3$	0
(41265), (235)	$-6(a-b)^3$	$6(a-b)^3$	$-6(a-b)^3$	0

Case 4 ($g = (123)(456)$). Note that $Z(g) = \langle (123), (456), (14)(25)(36) \rangle \times \text{Sym}_{\{7, \dots, n\}}$, and the subgroup

$$H = \langle (123), (456) \rangle$$

is normal in $Z(g)$. The factorizations of g as a product of a 5-cycle and a 3-cycle are in a single $Z(g)$ -orbit (of size 18), which partitions into two H -orbits (of size 9)

$$Z(g)((12345), (563)) = {}^H((12345), (563)) \cup {}^H((45612), (236)).$$

To show $\phi_g \equiv 0$, it suffices to verify

$$\sum_{(x,y) \in {}^H((12345), (563))} \phi_{x,y}(e_i, e_j, e_k) = 0$$

for H -orbit representatives $\{e_i, e_j, e_k\}$ of the basis triples. Since $V^x \cap V^y \subseteq \ker \phi_{x,y}$, we only consider $\{e_i, e_j, e_k\} \subseteq \{e_l \mid 1 \leq l \leq 6\}$. The remaining triples partition into H -orbits

$${}^H\{e_1, e_2, e_4\}, \quad {}^H\{e_1, e_4, e_5\}, \quad {}^H\{e_1, e_2, e_3\}, \quad \text{and} \quad {}^H\{e_4, e_5, e_6\}.$$

(The sum of orbit sizes is $9 + 9 + 1 + 1 = \binom{6}{3}$.) We simplify each term $\phi_{x,y}(e_i, e_j, e_k)$ of the sum and record the results in a table with rows indexed by the elements of $H((12345), (563))$. Each zero entry is color-coded and tagged by the applicable part of Lemma 7.6, all nonzero entries are found using Lemma 7.6 (3), and each column sum is zero.

x, y	$\phi_{x,y}(e_1, e_2, e_3)$	$\phi_{x,y}(e_4, e_5, e_6)$	$\phi_{x,y}(e_1, e_2, e_4)$	$\phi_{x,y}(e_1, e_4, e_5)$
(12345), (563)	0	0	0	0
(12356), (643)	0	0	0	0
(12364), (453)	0	0	0	$6(a-b)^3$
(23145), (561)	0	0	0	0
(23156), (641)	0	0	$6(a-b)^3$	0
(23164), (451)	0	0	$-6(a-b)^3$	0
(31245), (562)	0	0	0	0
(31256), (642)	0	0	$6(a-b)^3$	0
(31264), (452)	0	0	$-6(a-b)^3$	$-6(a-b)^3$

Case 5 ($g = (12345)$). Note that $Z(g) = H \times \text{Sym}_{\{6, \dots, n\}}$, where H is the normal subgroup

$$H = \langle (12345) \rangle.$$

The factorizations of g as a product of a 5-cycle and a 3-cycle are in three $Z(g)$ -orbits

$$Z(g)((12354), (354)), \quad Z(g)((12534), (254)), \quad \text{and} \quad Z(g)((12346), (645)),$$

which partition into H -orbits

$$\begin{aligned} Z(g)((12354), (354)) &= {}^H((12354), (354)), \\ Z(g)((12534), (254)) &= {}^H((12534), (254)), \quad \text{and} \\ Z(g)((12346), (645)) &= \bigcup_{r \geq 6} {}^H((1234r), (r45)). \end{aligned}$$

To show $\phi_g \equiv 0$, it suffices to verify

$$\begin{aligned} &\sum_{(x,y) \in {}^H((12354), (354))} \phi_{x,y}(e_i, e_j, e_k) \\ &+ \sum_{(x,y) \in {}^H((12534), (254))} \phi_{x,y}(e_i, e_j, e_k) \\ &+ \sum_{(x,y) \in {}^H((12346), (645))} \phi_{x,y}(e_i, e_j, e_k) = 0 \end{aligned}$$

for H -orbit representatives $\{e_i, e_j, e_k\}$ of the basis triples. Since $V^x \cap V^y \subseteq \ker \phi_{x,y}$, we only consider $\{e_i, e_j, e_k\} \subseteq \{e_l \mid 1 \leq l \leq 6\}$. The remaining triples partition into

H -orbits

$${}^H\{e_1, e_2, e_3\}, \quad {}^H\{e_1, e_2, e_4\}, \quad {}^H\{e_1, e_2, e_6\}, \quad \text{and} \quad {}^H\{e_1, e_3, e_6\}.$$

(The sum of orbit sizes is $5 + 5 + 5 + 5 = \binom{6}{3}$.) We simplify each term $\phi_{x,y}(e_i, e_j, e_k)$ of the sum and record the results in a table with rows indexed by the elements of $Z(g)((12354), (354)) \cup Z(g)((12534), (254)) \cup Z(g)((12346), (645))$. Each zero entry is color-coded and tagged by the applicable part of Lemma 7.6, all nonzero entries are found using Lemma 7.6 (3), and each column sum is zero.

x, y	$\phi_{x,y}(e_1, e_2, e_3)$	$\phi_{x,y}(e_1, e_2, e_4)$	$\phi_{x,y}(e_1, e_2, e_6)$	$\phi_{x,y}(e_1, e_3, e_6)$
$(12354), (354)$	0	0	0	0
$(23415), (415)$	0	$2(a-b)^3$	0	0
$(34521), (521)$	$-2(a-b)^3$	$2(a-b)^3$	0	0
$(45132), (132)$	0	$-2(a-b)^3$	0	0
$(51243), (243)$	$2(a-b)^3$	$-2(a-b)^3$	0	0
$(12534), (254)$	0	$-4(a-b)^3$	0	0
$(23145), (315)$	$4(a-b)^3$	0	0	0
$(34251), (421)$	$-4(a-b)^3$	0	0	0
$(45312), (532)$	$4(a-b)^3$	0	0	0
$(51423), (143)$	$-4(a-b)^3$	$4(a-b)^3$	0	0
$(12346), (645)$	0	0	0	0
$(23456), (651)$	0	0	$2(a-b)^3$	0
$(34516), (612)$	$-2(a-b)^3$	0	0	$-2(a-b)^3$
$(45126), (623)$	$2(a-b)^3$	0	$-2(a-b)^3$	$2(a-b)^3$
$(51236), (634)$	0	0	0	0

Case 6 ($g = (12)(34)$). Note that $Z(g) = H \times \text{Sym}_{\{5, \dots, n\}}$, where H is the dihedral group

$$H = \langle (1324), (12) \rangle = \{1, (1324), (12)(34), (4231), (12), (13)(24), (34), (14)(23)\}.$$

The factorizations of g as a product of a 5-cycle and a 3-cycle are in a single $Z(g)$ -orbit, which we partition into H -orbits

$$Z(g)((12345), (542)) = \bigcup_{r \geq 5} {}^H((1234r), (r42)).$$

To show $\phi_g \equiv 0$, it suffices to verify

$$\sum_{(x,y) \in {}^H((12345), (542))} \phi_{x,y}(e_i, e_j, e_k) = 0$$

for H -orbit representatives $\{e_i, e_j, e_k\}$ of the basis triples. Since $V^x \cap V^y \subseteq \ker \phi_{x,y}$, we only consider $\{e_i, e_j, e_k\} \subseteq \{e_l \mid 1 \leq l \leq 5\}$. The remaining triples partition into H -orbits

$${}^H\{e_1, e_2, e_3\}, \quad {}^H\{e_1, e_2, e_5\}, \quad \text{and} \quad {}^H\{e_1, e_3, e_5\}.$$

(The sum of the orbit sizes is $4 + 2 + 4 = \binom{5}{3}$.) We simplify each term $\phi_{x,y}(e_i, e_j, e_k)$ of the sum and record the results in a table with rows indexed by the elements of ${}^H((12345), (542))$. Notice that $\phi_{(21435), (531)}(e_1, e_3, e_5) = \phi_{(43215), (513)}(e_1, e_3, e_5) = 0$ follows from Lemma 7.6 (4), all other zero entries result from Lemma 7.6 (1), and all nonzero entries are found using Lemma 7.6 (3). Each column sum is zero.

x, y	$\phi_{x,y}(e_1, e_2, e_3)$	$\phi_{x,y}(e_1, e_2, e_5)$	$\phi_{x,y}(e_1, e_3, e_5)$
$(12345), (542)$	0	$-4(a-b)^3$	0
$(12435), (532)$	$4(a-b)^3$	$-4(a-b)^3$	$4(a-b)^3$
$(21345), (541)$	0	$4(a-b)^3$	$-4(a-b)^3$
$(21435), (531)$	$-4(a-b)^3$	$4(a-b)^3$	0
$(34125), (524)$	0	$-4(a-b)^3$	0
$(34215), (514)$	0	$4(a-b)^3$	$-4(a-b)^3$
$(43125), (523)$	$4(a-b)^3$	$-4(a-b)^3$	$4(a-b)^3$
$(43215), (513)$	$-4(a-b)^3$	$4(a-b)^3$	0

Case 7 ($g = (123)$). Note that $Z(g) = \langle (123) \rangle \times \text{Sym}_{\{4, \dots, n\}}$. The factorizations of g as a product of a 5-cycle and a 3-cycle are in a single $Z(g)$ -orbit

$$Z(g)((12345), (543)) = \{((123rs), (sr3)) \mid \{r, s\} \subseteq \{4, \dots, n\}\}.$$

Consider the subgroup $H = \langle (123), (45) \rangle$ of $Z(g)$. To show $\phi_g \equiv 0$, it suffices to verify

$$\sum_{(x,y) \in {}^H((12345), (543))} \phi_{x,y}(e_i, e_j, e_k) = 0$$

for H -orbit representatives $\{e_i, e_j, e_k\}$ of the basis triples. Since $V^x \cap V^y \subseteq \ker \phi_{x,y}$, we only consider $\{e_i, e_j, e_k\} \subseteq \{e_l \mid 1 \leq l \leq 5\}$. The remaining triples partition into H -orbits

$${}^H\{e_1, e_2, e_3\}, \quad {}^H\{e_1, e_2, e_4\}, \quad \text{and} \quad {}^H\{e_1, e_4, e_5\}.$$

(The sum of the orbit sizes is $1 + 6 + 3 = \binom{5}{3}$.) We simplify each term $\phi_{x,y}(e_i, e_j, e_k)$ of the sum and record the results in a table with rows indexed by the elements of ${}^H((12345), (543))$. Notice that $\phi_{(23154), (451)}(e_1, e_4, e_5) = \phi_{(23145), (541)}(e_1, e_4, e_5) = 0$ follows from Lemma 7.6 (4), all other zero entries result from Lemma 7.6 (1), and all

nonzero entries are found using Lemma 7.6 (3). Each column sum is zero.

x, y	$\phi_{x,y}(e_1, e_2, e_3)$	$\phi_{x,y}(e_1, e_2, e_4)$	$\phi_{x,y}(e_1, e_4, e_5)$
$(12354), (453)$	0	0	$2(a-b)^3$
$(12345), (543)$	0	0	$-2(a-b)^3$
$(23154), (451)$	0	$-2(a-b)^3$	0
$(23145), (541)$	0	$2(a-b)^3$	0
$(31254), (452)$	0	$-2(a-b)^3$	$-2(a-b)^3$
$(31245), (542)$	0	$2(a-b)^3$	$2(a-b)^3$

□

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DEPARTMENT OF MATHEMATICS AND STATISTICS, CALIFORNIA STATE POLYTECHNIC UNIVERSITY,
POMONA, CALIFORNIA 91768, USA

E-mail address: `brianaf@cupp.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, IDAHO STATE UNIVERSITY, POCA TELLO, IDAHO
83209, USA

E-mail address: `krilcath@isu.edu`